Introduction

Mechanics begins with a long tradition of qualitative investigation culminating with KEPLER and GALILEO. Following this is the period of quantitative theory (1687–1889) characterized by concomitant developments in mechanics, mathematics, and the philosophy of science that are epitomized by the works of NEWTON, EULER, LAGRANGE, LAPLACE, HAMILTON, and JACOBI. Both of these periods are thoroughly described in DÜGAS [1955].

Throughout these periods, the distinguished special case of *celestial mechanics* had a dominant role (see MOULTON [1902] for additional historical details). Formalized in the quantitative period as the *n-body problem*, it recurs in the writings of all of the great figures of the time. The question of *stability* was one of main concerns, and was analyzed with series expansion techniques by LAPLACE (1773), LAGRANGE (1776), POISSON (1809), DIRICHLET (1858), and HARETU (1878), all of whom claimed to have proved that the solar system was stable.

As DIRICHLET died before writing down this proof, KING OSCAR of Sweden offered a prize for its discovery, which was given to POINCARÉ in 1889. The results of POINCARÉ, suggesting that the series expansions of LAPLACE et al. diverged, and the discovery by BRUNS [1887] that no quantitative methods other than series expansions could resolve the n-body problem brought the quantitative period to an end. (See MOSER [1973a] for additional historical information.) For celestial mechanics this situation represented a great dilemma, comparable to the crises associated with relativity and quantum theory in other aspects of mechanics. The resolution we owe to the genius of POINCARÉ, who resurrected the qualitative point of view, accompanied by completely new mathematical methods. The inventions of POINCARÉ, culminating in modern differential geometry and topology, constitute a recent and lesser known example of concomitant development of mathematics and mechanics, comparable to calculus, differential equations, and variational theory.

The neoqualitative period in mechanics, that is, from POINCARÉ to the present, consists primarily in the amplification of the qualitative, geometric methods of POINCARÉ, the application of these methods to the qualitative questions of the previous period—for example, stability in the n-body problem—and the consideration of new qualitative questions that could not previously be asked.

POINCARÉ's methods are characterized first of all by the global geometric point of view. He visualized a dynamical system as a field of vectors on phase space, in which a solution is a smooth curve tangent at each of its points to the vector based at that point. The qualitative theory is based on geometrical properties of the *phase portrait*: the family of solution curves, which fill up the entire phase space. For questions such as stability, it is necessary to study the entire phase portrait, including the behavior of solutions for all values of the time parameter. Thus it was essential to consider the entire phase space at once as a geometric object. Doing so, POINCARÉ found the prevailing mathematical model for mechanics inadequate, for its underlying space was Euclidean, or a domain of several real variables, whereas for a mechanical problem with angular variables or constraints, the phase space might be a more general, nonlinear space, such as a generalized cylinder. Thus the global view in the qualitative theory led POINCARÉ to the notion of a *differentiable manifold* as the phase space in mechanics. In mechanical systems, this manifold always has a special geometric structure pertaining to the occurrence of phase variables in canonically conjugate pairs, called a *symplectic structure*. Thus the new mathematical model for mechanics consists of a *symplectic manifold*, together with a *Hamiltonian vector field*, or global system of first-order differential equations preserving the symplectic structure.

This model offers no natural system of coordinates. Indeed a manifold admits a coordinate system only locally, so it is most efficient to use the intrinsic calculus of CARTAN rather than the conventional calculus of NEWTON in the analysis of this model. The complete description of this model for mechanics comes quite a bit after POINCARÉ, as the intrinsic calculus was not fully developed until the 1940s. One advantage of this model is that by suppressing unnecessary coordinates the full generality of the theory becomes evident.

The second characteristic of the qualitative theory is the replacement of analytical methods by differential-topological ones in the study of the phase portrait. For many questions, for example the stability of the solar system, one is interested finally in qualitative information about the phase portrait. In earlier times, the only techniques available were analytical. By obtaining a complete or approximate quantitative solution, qualitative or geometric properties could be deduced. It was POINCARÉS idea to proceed directly to qualitative information by qualitative, that is, geometric methods. Thus POINCARÉ, BIRKHOFF, KOLMOGOROV, ARNOLD, and MOSER show the existence of periodic solutions in the three-body problem by applying differential-topological theorems to the phase portraits in addition to analytical methods. No analytical description of these orbits has been given. In some cases the orbits have been plotted approximately by computers, but of course the computer cannot prove that these solutions are periodic.

A third aspect of the qualitative point of view is a new question that emerges in it—the problem of *structural stability*, the most comprehensive of many different notions of stability. This problem, first posed in 1937 by Andronov–Pontriagin, asks: If a dynamical system X has a known phase portrait P, and is then perturbed to a slightly different system X' (for example, changing the coefficients in its differential equation slightly), then is the new phase portrait P' close to P in some topological sense? This problem is of obvious importance, since in practice the qualitative information obtained for P is to be applied not to X, but to some nearby system X', because the coefficients of the equation may be determined experimentally or by an approximate model and therefore approximately.

The traditional mutuality of mechanics and philosophy has declined in recent years, perhaps because of the justifiable interest in the problems posed by relativity and quantum theory. But current problems in mechanics give new insight into the structure of physical theories.

At the turn of this century a simple description of physical theory evolved, especially among continental physicists—DUHEM, POINCARÉ, MACH, EINSTEIN, HADAMARD, HILBERT—which may still be quite close to the views of many mathematical physicists. This description—most clearly enunciated by DUHEM [1954]—consisted of an *experimental domain*, a *mathematical model*, and a *conventional interpretation*. The model, being a mathematical system, embodies the logic, or axiomatization, of the theory. The interpretation is an agreement connecting the parameters and therefore the conclusions of the model and the observables in the domain.

Traditionally, the philosopher-scientists judge the usefulness of a theory by the criterion of *adequacy*, that is, the verifiability of the predictions, or the quality of the agreement between the interpreted conclusions of the model and the data of the experimental domain. To this DUHEM adds, in a brief example [1954, pp. 138 ff.], the criterion of *stability*.

This criterion, suggested to him by the earliest results of qualitative mechanics (HADAMARD), refers to the stability or continuity of the predictions, or their adequacy, when the model is slightly perturbed. The general applicability of this type of criterion has been suggested by RENÉ THOM [1975].

This stability concerns variation of the model only, the interpretation and domain being fixed. Therefore, it concerns mainly the model, and is primarily a mathematical or logical question. It has been studied to some extent in a general logical setting by the physicologicians BOULIGAND [1935] and DESTOUCHES [1935], but probably it is safe to say that a clear enunciation of this criterion in the correct generality has not yet been made. Certainly all of the various notions of stability in qualitative mechanics and ordinary differential equations are special cases of this notion, including LAPLACE's problem of the stability of the solar system and structural stability, as well as THOM's stability of biological systems.

Also, although this criterion has not been discussed very explicitly by physicists, it has functioned as a tacit assumption, which may be called the *dogma of stability*. For example, in a model with differential equations, in which stability may mean structural stability, the model depends on parameters, namely the coefficients of the equation, each value of which corresponds to a different model. As these parameters can be determined only approximately, the theory is useful only if the equations are structurally stable, which cannot be proved at present in many important cases. Probably the physicist must rely on faith at this point, analogous to the faith of a mathematician in the consistency of set theory.

An alternative to the dogma of stability has been offered by THOM [1975]. He suggests that stability, precisely formulated in a specific theory, be added to the model as an additional hypothesis. This formalization, despite the risk of an inconsistent axiomatic system, reduces the criterion of stability to an aspect of the criterion of adequacy, and in addition may admit additional theorems or predictions in the model. As yet no implications of this axiom are known for celestial mechanics, but THOM has described some conclusions in his model for biological systems.

A careful statement of this notion of stability in the general context of physical theory and epistemology could be quite useful in technical applications of mechanics as well as in the formulation of new qualitative theories in physics, biology, and the social sciences.

Most of this book is devoted to a precise statement of mathematical models for mechanical systems and to precise definitions of various types of stability in this narrow context. These are illustrated by a number of examples, but by one example in depth, namely, the restricted three-body problem in Chapter 10.