

Introduction

The theory of Eisenstein series is one of the fundamental tools for the study of automorphic forms and an indispensable part of the Langlands program. It determines the continuous spectrum of the group as well as all the non-cuspidal representations appearing discretely in the spectrum as residues of these series which are meromorphic functions of several variables. This was the subject matter of Langlands' groundbreaking work in [La2].

While the work in [La2] is profound and complicated, it has its roots in the classical Eisenstein series on the upper half plane \mathfrak{h} . It covers both the holomorphic and real analytic cases [Iw] and mainly builds modular forms on the corresponding Riemannian symmetric spaces which account for the continuous part of their spectrums. In the classical setting it accounts for the continuous spectrum for Riemann surfaces $\Gamma \backslash \mathfrak{h}$, as Γ ranges over congruence subgroups of $SL_2(\mathbb{Z})$.

It is instructive to consider the case of $\Gamma = SL_2(\mathbb{Z})$, where $E(z, s)$ is the Eisenstein series

$$E(z, s) = \sum_{\gamma \in B_2(\mathbb{Z}) \backslash SL_2(\mathbb{Z})} \text{Im}(\gamma z)^s$$

defined at the cusp at infinity, where $s \in \mathbb{C}$, $z \in \mathfrak{h}$ and

$$B_2(\mathbb{Z}) = \left\{ \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{Z} \right\}.$$

One has the Fourier expansion at infinity given by (cf. [Iw])

$$\begin{aligned} E(z, s) &= y^s + \pi^{1/2} \frac{\Gamma(\frac{2s-1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)} y^{1-s} \\ &\quad + \pi^s \Gamma(s)^{-1} \zeta(2s)^{-1} \sum_{n \neq 0} |n|^{-1/2} \left(\sum_{ab=|n|} \left(\frac{a}{b}\right)^{s-\frac{1}{2}} \right) W_s(nz), \end{aligned}$$

where $z = x + iy$ and $W_s(z)$ is the Whittaker function defined by

$$W_s(x + iy) = 2y^{1/2} K_{s-1/2}(2\pi y) \exp(2\pi i x),$$

with $K_\nu(z)$ the K -Bessel function

$$K_\nu(z) = \int_0^\infty e^{-z \cosh t} \cosh(\nu t) dt,$$

where $\operatorname{Re}(z) > 0$ and $\operatorname{Re}(\nu) > -1/2$ (cf. [Iw]).

We note that with a suitable normalization of s the coefficient of y^{1-s} , the main part of constant term of the Fourier expansion, is

$$L(2s - 1)/L(2s)$$

where

$$L(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

is the completed L -function for the Riemann zeta function $\zeta(s)$.

On the other hand the non-constant Fourier coefficients of $E(z, s)$ are basically the inverse

$$1/L(2s)$$

of the L -function $L(2s)$ which appears and controls the constant term via the quotient

$$L(2s - 1)/L(2s).$$

We note that the poles of $E(z, s)$ as a function of s are basically controlled by those of $L(2s - 1)$ and are precisely those of the constant term.

This phenomenon is quite general and in fact constant terms of general Eisenstein series are ratios of L -functions, while their non-constant Whittaker Fourier coefficients are inverses of these L -functions and here lies the point of inception of this book.

The first computation of the constant term in any generality appeared in Langlands' monograph *Euler Products*, which appeared in 1971 [La1]. It led him to the definition of L -groups and L -functions which are the foundations of his fundamental and deep conjectures in [La3]¹. Moreover, in a letter to Godement, Langlands posed a number of problems and directions which may be considered as the starting point of the so-called Langlands-Shahidi method which, besides its applications to automorphic forms and representation theory, has been one of the two principal tools, the second being the converse theorems of Cogdell and Piatetski-Shapiro, in establishing recent surprising and new cases of the Langlands Functoriality Conjecture, the main driving principle in the Langlands program which has allowed a considerable sharpening of estimates towards the Ramanujan and Selberg conjectures.

The Langlands-Shahidi method is that of studying L -functions that appear in the constant terms of Eisenstein series as we explained in the above example, mainly when the inducing representations are generic [CS, S, Sh3, Sh7, Sh8]. A better understanding of this theory has been yielding important results not only in the theory of automorphic forms and number theory but in representation theory and harmonic analysis over local fields as well [CSh, Sh8]. Some of the conjectures rooted in this method, such as

¹This agrees well with the fact that [La1] was in fact based on his 1967 James K. Whittemore Lectures at Yale.

the tempered L -packet conjecture which demands the existence of a generic member in each such packet, have been finding applications in various parts of the theory. In particular, one expects that the theory of L -functions attached to generic forms should extend to all others via this conjecture and the basic properties of L -packets [Sh6, Sh8, V]. We refer to [Sh23] for a brief account of the development of this method in other contexts.

This book is a detailed and more complete version of an earlier manuscript which itself came to existence after a course the author taught at Caltech during the Spring Quarter of 1997 which was later repeated at Purdue several times. The present version is a rather general treatment of the method, covering a much broader ground, and has been presented as a three semester course at Purdue, ending in the Fall semester of 2008.

The book presents the method, practically from its beginning, although not in the chronological order in which it was developed. The first two chapters are devoted to generalities about reductive groups, their L -groups, the basic representation theory needed for what follows, both local and global, and the basic properties of automorphic L -functions. We have tried to define many of these objects and state their relevant properties, some with proofs and some with no proofs or just sketches. These two chapters then may be considered as preliminaries for what follows.

Chapters 3 to 9 are devoted to developing the method. With a few exceptions, complete proofs are given for the material in Chapters 3, 4 and 5. They discuss the main local tools of the method: Whittaker models, intertwining operators and local coefficients. Among the material covered in Chapter 3 is a more or less complete proof of the Casselman–Shalika formula [CS], one of the method’s central tools.

In Chapter 6, while the basic properties of Eisenstein series are assumed, by referring to standard sources, calculation of constant terms, their connection with intertwining operators and L -functions, as well as the basic induction of the method, are included in some detail.

Chapter 7 discusses the non-constant terms of the Eisenstein series and includes their calculation and their consequences in non-vanishing of L -functions.

Functional equations are proved in Chapter 8. The chapter includes explicit and precise results and formulas for local coefficients. Here we define some of the general notions that are central to the method, such as “multiplicativity”, and discuss some examples. On the other hand the proof of the main theorem of the chapter, Theorem 8.3.2, as well as all the calculations of local coefficients, are left to references as they are rather involved.

It is in Chapter 8 that we define, rather carefully, the local L -functions and root numbers at every prime using our method. We should mention that this is a central issue, since a number of cases of local Langlands correspondence require equalities of L -functions and root numbers of Artin with those defined here [HT, He2, JiSo, La4, La5, Sh6, Sh23]. We conclude the chapter by discussing some of the connections of our method with local

harmonic analysis and representation theory. In particular we discuss the normalization of intertwining operators by means of our factors as suggested by Langlands [La2]. This has played a crucial role in many applications of this method to representation theory.

Chapter 9 covers further analytic properties of these L -functions such as finiteness of poles as well as their boundedness in vertical strips [GSh2]. The holomorphy under highly ramified twists [K1, KSh2] is stated and proved in Section 9.3.

The material in Section 9 is based on Assumptions 9.1.6 and 9.1.7. As we point out in Remark 9.1.1, it appears that the validity of these assumptions can now be proved in full generality following a recent manuscript of Heiermann and Opdam [HO], mixed with the work of Kim and others [As, CSh, K3, KKi, Ki, MuSh, Sh8]. One can therefore safely assume that the results in Section 9 do not require restrictive assumptions. Moreover, the same manuscript implies the validity of Conjecture 8.4.3 in general. Thus every L -function attached to a tempered representation defined from our method is holomorphic for $\text{Re}(s) > 0$. This is always the case for an Artin L -function.

The last chapter is concerned with applications to functoriality and discusses a number of recent cases of functoriality proved by Kim and the author [KSh2, K4], as well as the author's joint results with Cogdell, Kim and Piatetski-Shapiro, and the work of Kim and Krishnamurthy on unitary groups [CKPSS1, CKPSS2, CPSS3, KK]. While only sketches of the proofs are given, many basic definitions and notions such as symmetric power representations and L -functions are carefully defined here.

There are appendices supplementing several chapters, and an important concluding appendix which consists of tables of Dynkin diagrams that appeared in [La1] and [Sh7]. They have all been reformatted by William Casselman, who has kindly allowed us to include them here. In particular, he has reformatted the diagrams for the cases of quasisplit groups in [Sh7] to agree with those of Tits [Ti].

Here is a list of a number of expository articles on the subject by different authors [C1, C2, CPS3, GMi, GSh2, K5, Mur, Sh13, Sh17, Sh19, Sh20].

It is hoped that the reader will agree that to produce detailed proofs of every result stated here is beyond the scope of this book. On the other hand we have tried to provide ample and precise references to what is missing. Moreover, we have included a number of examples, some with full details, as well as a number of exercises.

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