Notation

S_ω	the projective connection attached to a symmetric bidifferential ω (see (1.41)),
T_X	the holomorphic (or regular) tangent bundle of a complex manifold (or smooth projective variety) X,
Θ_X	the sheaf of holomorphic vector fields of a complex manifold or a smooth projective variety X,
Ω_X^k	the sheaf of holomorphic k-forms of a complex manifold or a smooth projective variety X,
$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\xi)) \oplus \mathbb{C} \cdot c$,	
$\widehat{\mathfrak{g}}_N = \mathfrak{g} \otimes_{\mathbb{C}} \left(\bigoplus_{j=1}^N \mathbb{C}((\xi_j)) \right) \oplus \mathbb{C} \cdot c$,	
Δ	the root system of a complex simple Lie algebra and its Cartan subalgebra $(\mathfrak{g}, \mathfrak{h})$,
Δ_+	the set of positive roots,
θ	the longest root of a complex simple Lie algebra $\mathfrak{g}$,
$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$,	
$(\ ,\ )$	Cartan-Killing form of a complex simple Lie algebra $\mathfrak{g}$ normalized as $(\theta, \theta) = 2$,
P_+	the set of integral dominant weights of a complex simple Lie algebra $\mathfrak{g}$,
$P_\ell = \{\lambda \in P_+ \mid (\lambda, \theta) \leq \ell\}$,	
$\mu^\dagger = -w(\mu)$	for $\mu \in P_\ell$ where w is the longest element of the Weyl group of a complex simple Lie algebra $\mathfrak{g}$,
$\kappa = g^* + \ell$	where g^* is the dual Coxeter number of a complex simple Lie algebra $\mathfrak{g}$, ($\kappa = n + 1 + \ell$ in case $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$),
$c_v = \frac{\ell \dim \mathfrak{g}}{\kappa}$,	
$\Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2\kappa}$,	
$\widehat{\Delta}_v = \Delta_\lambda + \Delta_{\mu_1} - \Delta_{\mu_2}$,	
$\widehat{\Delta}_{\mathbf{v}} = \Delta_\lambda + \Delta_{\mu_1} - \Delta_{\mu_2}$ for a vertex $\mathbf{v} = \binom{\ \lambda\ }{\mu_1\ \mu_2}$,	
$X_n = \mathbb{C}^n \setminus \cup_{i<j} \Delta_{ij}, \quad \Delta_{ij} = \{(z_n, \ldots, z_1) \in \mathbb{C}^n \mid z_i = z_j\}$,	
$\mathcal{R}_n = \{\ (z_n, \ldots, z_1) \in \mathbb{C}^n \mid \lvert z_n \rvert > \lvert z_{n-1} \rvert > \cdots > \lvert z_1 \rvert \}$,	

CHAPTER 1

Riemann Surfaces and Stable Curves

1.1 Compact Riemann surfaces

1.1.1 Compact Riemann surfaces. A connected compact one-dimensional complex manifold is called a compact Riemann surface or a closed Riemann surface. In these Lecture Notes we shall mainly consider compact Riemann surfaces; we often omit "compact".

A compact Riemann surface R is an oriented closed surface, so it is diffeomorphic to the surface of a doughnut with g holes. The number g is called the *genus* of the Riemann surface R and often denoted by $g(R)$.

For a compact Riemann surface R of genus g its first homology group $H_1(R,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $2g$. Moreover, the intersection pairing $\alpha\cdot\beta$, $\alpha,\beta\in H_1(R,\mathbb{Z})$ is skew-symmetric and a bilinear mapping

$$H_1(R,\mathbb{Z})\times H_1(R,\mathbb{Z})\to\mathbb{Z}.$$

We can always find a symplectic basis $\{\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g\}$ of $H_1(R,\mathbb{Z})$, that is,

$$\alpha_i\cdot\alpha_j=0,\qquad \beta_i\cdot\beta_j=0,\quad \alpha_i\cdot\beta_j=\delta_{ij}.$$

Moreover, there are g linearly independent holomorphic one-forms on R. Let Θ_R be the sheaf of germs of holomorphic vector fields on R and ω_R be its dual. The sheaf ω_R is called the *canonical* sheaf. It is the sheaf associated with a complex line bundle K_X called the *canonical line bundle* of R. It is well-known that $H^0(R,\mathcal{O}_R(L))$ gives a holomorphic embedding of R into a complex projective space $\mathbb{P}^{2g-3}$ if the degree of the line bundle L on R is at least $2g+1$. Hence all compact Riemann surfaces have the structure of a projective variety. In the following sometimes we regard a compact Riemann surface as a projective variety.

1.1.2 Complex analytic families of Riemann surfaces. In the following, for a complex manifold M and a point Q on M, by $T_Q(M)$ we mean the holomorphic

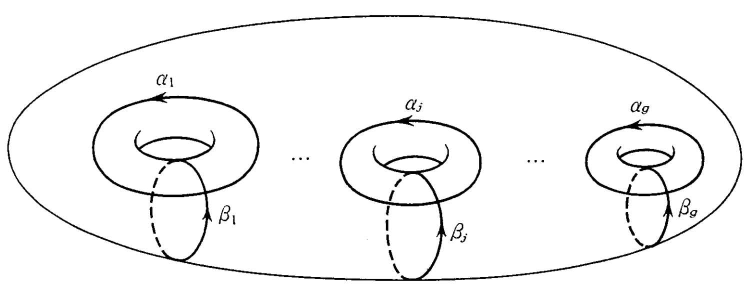

Figure 1.1 Riemann surface of genus g.

tangent vector space at the point Q. Also we assume that a complex manifold is connected, unless otherwise explicitly mentioned.

A holomorphic mapping $\pi\colon \mathcal{C} \to \mathcal{W}$ from an $(m+1)$-dimensional complex manifold $\mathcal{C}$ to an m-dimensional complex manifold $\mathcal{W}$ satisfying the following conditions is called a *smooth family of compact Riemann surfaces* or *a complex analytic family of compact Riemann surfaces* over the complex manifold $\mathcal{W}$.

(1) The mapping π is proper. That is, for any compact set K of the complex manifold $\mathcal{W}$, the inverse image $\pi^{-1}(K)$ is compact.
(2) The mapping π is smooth and holomorphic. That is, for any point $P \in \mathcal{C}$ the linear mapping $(d\pi)_P\colon T_P(\mathcal{C}) \to T_{\pi(P)}(\mathcal{W})$ of the holomorphic tangent spaces is surjective.
(3) For any point $w \in \mathcal{W}$ the fibre $\pi^{-1}(w)$ is connected.

By conditions (1) and (2) the fibre $C_w = \pi^{-1}(w)$ over each point $w \in \mathcal{W}$ is a compact Riemann surface. Also, for a point $w_0 \in \mathcal{W}$ we call $C_w,\ w \in \mathcal{W}$, a *deformation* of the compact Riemann surface C_{w_0}.

When a complex analytic family $\pi\colon \mathcal{C} \to \mathcal{W}$ of compact Riemann surfaces and a holomorphic mapping $h\colon \mathcal{S} \to \mathcal{W}$ are given, the fibre product $\mathcal{C} \times_{\mathcal{W}} \mathcal{S} \to \mathcal{S}$ is a complex analytic family over $\mathcal{S}$. We call the family $\mathcal{C} \times_{\mathcal{W}} \mathcal{S} \to \mathcal{S}$ the *pullback family* by the holomorphic mapping h.

Example 1.1 On the product $H \times \mathbb{C}$ of the upper half-plane $H = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ and the complex plane $\mathbb{C}$, the action of the group $\mathbb{Z}^2$ is defined by

$$\begin{array}{rccl} (m,n)\colon & H \times \mathbb{C} & \longrightarrow & H \times \mathbb{C},\ (m,n) \in \mathbb{Z}^2, \\ & (\tau, \zeta) & \longmapsto & (\tau, \zeta + m + n\tau). \end{array}$$

Since the action is fixed point free and properly discontinuous, the quotient space $\mathcal{E} = H \times \mathbb{C}/\mathbb{Z}^2$ has the structure of a two-dimensional complex manifold. The point $(\tau, \zeta) \in H \times \mathbb{C}$ determines a point $(\tau, [\zeta])$ of $\mathcal{E}$. The natural projection $p\colon H \times \mathbb{C} \to H$ induces a holomorphic mapping

$$\begin{array}{rccl} \pi\colon & \mathcal{E} & \longrightarrow & H, \\ & (\tau, [\zeta]) & \longmapsto & \tau. \end{array}$$

The holomorphic mapping $\pi\colon \mathcal{E} \to H$ satisfies the above conditions (1), (2), (3), hence is a complex analytic family of compact Riemann surfaces over H. For a point $\tau \in H$ the fibre $E_\tau = \pi^{-1}(\tau)$ is an elliptic curve (a one-dimensional complex torus) with fundamental periods $(1, \tau)$.

Example 1.2 The group $SL(2, \mathbb{R})$ operates on the upper half-plane H by

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$

For a positive integer n the normal subgroup $\Gamma(n)$ of $SL(2, \mathbb{Z})$ is defined by

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \;\middle|\; \begin{array}{ll} a \equiv d \equiv 1 & (\mathrm{mod}\ n) \\ b \equiv c \equiv 0 & (\mathrm{mod}\ n) \end{array} \right\}.$$

Then, it is known that for $n \geq 3$ the action of the group $\Gamma(n)$ has no fixed points on H. Moreover, the group $\Gamma(n)$ acts on $\mathcal{E}$ of Example 1.1 by

$$(\tau, [\zeta]) \longmapsto \left(\frac{a\tau + b}{c\tau + d}, \left[\frac{\zeta}{c\tau + d} \right] \right).$$

The action is fixed point free. The holomorphic mapping $\pi\colon \mathcal{E} \to H$ is compatible with the actions of $\Gamma(n)$ on $\mathcal{E}$ and H. Hence we have a holomorphic mapping of the quotient spaces $\pi(n)\colon \mathcal{E}(n) =: \mathcal{E}/\Gamma(n) \to C(n) := H/\Gamma(n)$.

For $n \geq 3$, $\pi(n)\colon \mathcal{E}(n) \to C(n)$ is also a complex analytic family of compact Riemann surfaces.

1.1.3 Kodaira-Spencer mapping. Let us study a complex analytic family $\pi\colon \mathcal{C} \to \mathcal{W}$ of compact Riemann surfaces by using local coordinates. Choose a coordinate neighbourhood U of a point $0 \in \mathcal{W}$ and local coordinates $(w^1, \dots, w^m)$ of U with center 0. Let $\pi^{-1}(U)$ be covered by open sets $\{U_\lambda\}_{\lambda\in\Lambda} : \bigcup_{\lambda\in\Lambda} U_\lambda = \pi^{-1}(U)$. Since π is a smooth holomorphic mapping, we can choose local coordinates of U_λ as $(w^1, \dots, w^m,\ z_\lambda)$. (That is, $w^1, \dots, w^m$ is chosen as a part of the local coordinates of U_λ for all λ.) If $U_\lambda \cap U_\mu \neq \emptyset$, between the two local coordinates we have the following relation:

$$z_\lambda = f_{\lambda\mu}\big(z_\mu, w^1, \dots, w^m\big). \tag{1.1}$$

Here, $f_{\lambda\mu}$ is holomorphic on $U_\lambda \cap U_\mu$. We may regard the complex manifold $\pi^{-1}(U)$ as obtained by patching $\{U_\lambda\}_{\lambda\in\Lambda}$ together by the relation (1.1). Then $(w^1, \dots, w^m)$ are regarded as parameters of changing the patching. That is, for $(w^1, \dots, w^m) = (0, \dots, 0)$ the manifold $\pi^{-1}\big((0, \dots, 0)\big)$ is a compact Riemann surface C_0, and it is obtained by patching by $z_\lambda = f_{\lambda\mu}\big(z_\mu, 0, \dots, 0\big)$. And for $w = (w^1, \dots, w^m)$ we obtain $C_w = \pi^{-1}(w)$ by changing the patching of C_0 slightly by w. Hence, the degree one term of the Taylor expansion of (1.1) with respect to the variables $w^1, \dots, w^m$ gives the first order approximation of deformations (usually called *infinitesimal deformation*) of the complex manifold C_0. This means that if $U_\lambda \cap U_\mu \neq \emptyset$, then

$$\left\{ \frac{\partial f_{\lambda\mu}}{\partial w^k}(z_\mu, 0, \dots, 0) \right\}, \quad 1 \leq k \leq m,$$

give information on the deformation of the compact Riemann surface C_0.

For $U_\lambda \cap U_\mu \neq \emptyset$, let us consider the holomorphic tangent vector field

$$\theta^{(k)}_{\lambda\mu} = \frac{\partial f_{\lambda\mu}}{\partial w^k}\big(z_\mu, 0, \dots, 0\big)\frac{\partial}{\partial z_\lambda} \tag{1.2}$$

on $C_0 \cap U_\lambda \cap U_\mu$. Note that in the above definition of the vector field $\theta^{(k)}_{\lambda\mu}$ we use the vector field $\dfrac{\partial}{\partial z_\lambda}$ on U_λ, but not $\dfrac{\partial}{\partial z_\mu}$. If we have $U_\lambda \cap U_\mu \cap U_\nu \neq \emptyset$, then by (1.1) we have

$$\begin{aligned} z_\lambda &= f_{\lambda\mu}\left(f^1_{\mu\nu}(z_\nu, w), w^1, \dots, w^m\right) \\ &= f_{\lambda\nu}(z_\nu, w^1, \dots, w^m). \end{aligned}$$

Hence, on $U_\lambda \cap U_\mu \cap U_\nu$ we have

$$\begin{aligned}
\theta^{(k)}_{\lambda\nu} &= \frac{\partial f_{\lambda\nu}}{\partial w^k}(z_\nu, 0)\frac{\partial}{\partial z_\lambda} \\
&= \frac{\partial f_{\lambda\mu}}{\partial z_\mu}\big(f_{\mu\nu}(z_\nu,0),0\big)\cdot\frac{\partial f_{\mu\nu}}{\partial w^k}(z_\nu,0)\frac{\partial}{\partial z_\lambda} \\
&\qquad + \frac{\partial f_{\lambda\mu}}{\partial w^k}\big(f_{\mu\nu}(z_\nu,0),0\big)\frac{\partial}{\partial z_\lambda} \\
&= \frac{\partial f_{\mu\nu}}{\partial w^k}(z_\nu,,0)\left(\frac{\partial f_{\lambda\mu}}{\partial z_\mu}\frac{\partial}{\partial z_\lambda}\right) + \theta^{(k)}_{\lambda\mu} \\
&= \frac{\partial f_{\mu\nu}}{\partial w^k}(z_\nu, 0)\frac{\partial}{\partial z_\mu} + \theta^{(k)}_{\lambda\mu} \\
&= \theta^{(k)}_{\mu\nu} + \theta^{(k)}_{\lambda\mu}.
\end{aligned}$$

That is, $\left\{\theta^{(k)}_{\lambda\mu}\right\}$ is a Čech one-cocycle with coefficients in holomorphic tangent vector fields. The cohomology class of $H^1(C_0, \Theta)$ defined by the above one-cocycle is denoted by the same symbol $\left\{\theta^{(k)}_{\lambda\mu}\right\}$. Here, by Θ we denote the sheaf of germs of holomorphic vector fields on C_0.

The cohomology class $\left\{\theta^{(k)}_{\lambda\mu}\right\}$ is uniquely determined if we fix local coordinates $(w^1, \ldots, w^m)$, and is independent of the choice of open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of $\pi^{-1}(U)$ by coordinate neighbourhoods and local coordinates $(z_\lambda, w^1, \ldots, w^m)$. This is proved by direct calculations. Therefore, we can define a linear mapping from the holomorphic tangent vector space $T_0\mathcal{W}$ at $0 \in \mathcal{W}$ to $H^1(C_0, \Theta)$:

$$\begin{array}{rccc}
\rho_0\colon & T_0\mathcal{W} & \longrightarrow & H^1(C_0,\Theta), \\
& \sum a_k \frac{\partial}{\partial w^k} & \longmapsto & \sum a_k \left\{\theta^{(k)}_{\lambda\nu}\right\}.
\end{array} \tag{1.3}$$

This linear mapping is called the *Kodaira-Spencer mapping*.

By the way, in the definition of $\left\{\theta^{(k)}_{\mu\nu}\right\}$ in (1.2), if, instead of putting $w = 0$, we put

$$\theta^{(k)}_{\lambda\mu}(w) = \frac{\partial f_{\lambda\mu}}{\partial w^k}(z_\mu, w)\frac{\partial}{\partial z_\lambda},$$

then by the same argument as above we can show that $\left\{\theta^{(k)}_{\lambda\mu}(w)\right\} \in H^1(C_w, \Theta_{C_w})$. Moreover, $\theta^{(k)}_{\lambda\mu}(w)$ is holomorphic in w. Hence, similarly to (1.3) we obtain the $\mathcal{O}_W$-homomorphism

$$\begin{array}{rcc}
\rho\,:\,\Theta_{\mathcal{W}} & \longrightarrow & R^1\pi_*\Theta_{\mathcal{C}/\mathcal{W}}, \\
\sum a_k(w)\frac{\partial}{\partial w^k} & \longrightarrow & \sum a_k(w)\left\{\theta^{(k)}_{\lambda\mu}(w)\right\};
\end{array} \tag{1.4}$$

here, $\Theta_{\mathcal{C}/\mathcal{W}}$ is the sheaf of germs of relative holomorphic tangent vector fields of $\pi\colon \mathcal{C} \to \mathcal{W}$. The $\mathcal{O}_{\mathcal{W}}$-homomorphism ρ is also called the Kodaira-Spencer mapping.

Since the complex analytic family $\pi\colon \mathcal{C} \to \mathcal{W}$ is a smooth holomorphic mapping, there exists an exact sequence of $\mathcal{O}_{\mathcal{C}}$-homomorphisms

$$0 \longrightarrow \Theta_{\mathcal{C}/\mathcal{W}} \longrightarrow \Theta_{\mathcal{C}} \longrightarrow \pi^*\Theta_{\mathcal{W}} \longrightarrow 0. \tag{1.5}$$

From the exact sequence we obtain a long exact sequence

$$\longrightarrow \pi_*\Theta_{\mathcal{C}} \longrightarrow \pi_*\pi^*\Theta_{\mathcal{W}} = \Theta_{\mathcal{W}} \xrightarrow{\tau} R^1\pi_*\Theta_{\mathcal{C}/\mathcal{W}} \longrightarrow R^1\pi_*\Theta_{\mathcal{C}}.$$

It is easily seen that by the definition of ρ the Kodaira-Spencer mapping (1.4) is nothing but the $\mathcal{O}_{\mathcal{W}}$-homomorphism $\Theta_{\mathcal{W}} \xrightarrow{\tau} R^1\pi_*\Theta_{\mathcal{C}/\mathcal{W}}$ of the long exact sequence.

Remark 1.3 Similarly we can define a complex analytic family of compact complex manifolds and the Kodaira-Spencer mapping.

Let us calculate the Kodaira-Spencer mapping of the family of elliptic curves constructed in Example 1.2.

Example 1.4 Let us first consider an elliptic curve $E = \mathbb{C}/(1, \tau_0)$ $\tau_0 \in H$. Let z be a global coordinate of the affine space $\mathbb{C}$ and o the origin of the elliptic curve E. From the exact sequence

$$0 \to \Theta_E \to \Theta_E(*o) \xrightarrow{t} z^{-1}\mathbb{C}[z^{-1}]\frac{d}{dz} \to 0$$

we have the exact sequence

$$0 \to H^0(E, \Theta_E(*o)) \xrightarrow{t} z^{-1}\mathbb{C}[z^{-1}]\frac{d}{dz} \xrightarrow{\theta} H^1(E, \Theta_E) \to 0$$

where we put

$$\begin{aligned} \Theta_E(*o) &= \varinjlim_{m\to\infty} \Theta(mo) \\ H^0(E, \Theta_E(*o)) &= \varinjlim_{m\to\infty} H^0(E, \Theta(mo)). \end{aligned}$$

Note that for the Weierstrass $\wp$-function, we have

$$\frac{(-1)^n}{n!}\wp(z)\frac{d}{dz} \in H^0(E, \mathcal{O}_E((n+2)o)) \setminus H^0(E, \mathcal{O}_E((n+1)o))$$

for all non-negative integers n. Hence, $z^{-1}\dfrac{d}{dz}$ has non-zero image $\theta(z^{-1}\dfrac{d}{dz})$ in $H^1(E, \Theta_E)$. Let us describe the image. Let $\{\mathcal{U}_\lambda\}_{\lambda=0}^N$ be a small open covering of E such that $\mathcal{U}_0$ is a coordinate neighbourhood of o with local coordinate z and such that

$$o \notin \mathcal{U}_0 \cap \mathcal{U}_\mu, \quad \mu \geq 1.$$

Then the image of $z^{-1}\dfrac{d}{dz}$ is given by the cocycle

$$\tau_{\lambda\mu} = \begin{cases} z^{-1}\dfrac{d}{dz}, & \lambda = 0, \quad \mu \geq 1, \quad \mathcal{U}_{\lambda\mu} \neq \emptyset \\ -z^{-1}\dfrac{d}{dz}, & \lambda \geq 1, \quad \mu = 0, \quad \mathcal{U}_{\lambda\mu} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Let $p : \mathcal{E} \to H$ be the versal family constructed in Example 1.2. As we put $E = \mathbb{C}/(1, \tau_0)$, by using the above covering, we find an open covering $\{\widehat{\mathcal{U}}_\lambda\}_{\lambda=0}^N$ of $p^{-1}(V)$, where V is an open neighbourhood of τ_0 such that $p^{-1}(\tau_0) \cap \widehat{\mathcal{U}}_\lambda = \mathcal{U}_\lambda$. Let $(\tau_\lambda, z_\lambda)$ be local coordinates of $\widehat{\mathcal{U}}_\lambda$ such that

$$\begin{aligned} \tau_\lambda &= \tau - \tau_0 \\ z_0 &= z \\ z_\lambda &= z_\mu + m_{\lambda\mu} + n_{\lambda\mu}\tau \end{aligned}$$

with $m_{\lambda\mu}$, $n_{\lambda\mu} \in \mathbb{Q}$. Then, the Kodaira-Spencer class $\rho_{\tau_0}(\frac{\partial}{\partial\tau}) \in H^1(E, \Theta_E)$ is given by the cocycle

$$\theta_{\lambda\mu} = \{n_{\lambda\mu}\frac{d}{dz_\lambda}\}.$$

Let us consider the Dolbeault representations of the cocycles $\{\tau_{\lambda\mu}\}$ and $\{\theta_{\lambda\mu}\}$.

First consider the cocycle $\{\theta_{\lambda\mu}\}$. Put

$$\psi_\lambda = \frac{(\bar{z}_\lambda - z_\lambda)}{2\sqrt{-1}\mathrm{Im}\,\tau}\frac{d}{dz_\lambda}.$$

Then we have

$$\psi_\mu - \psi_\lambda = n_{\lambda\mu}\frac{d}{dz_\lambda} = \theta_{\lambda\mu}.$$

Therefore

$$\overline{\partial}\psi_\mu = \overline{\partial}\psi_\lambda = (2\sqrt{-1}\mathrm{Im}\,\tau)^{-1}\frac{d}{dz}d\bar{z}$$

is the corresponding Dolbeault representative of $\{\theta_{\lambda\mu}\}$.

Next consider the cohomology class $\{\tau_{\lambda\mu}\}$. We may assume that

$$\{\ z \in \mathbb{C} \mid |z| \leq \epsilon\} \subset \mathcal{U}_0$$

and

$$\{\ z \in \mathbb{C} \mid |z| \leq \epsilon/2\} \subset \mathcal{U}_0 \setminus \bigcup_{\lambda=1}^{N} \mathcal{U}_\lambda.$$

Let $f(z)$ be a real-valued C^∞ function in $\mathcal{U}_0$ such that $0 \leq f(z) \leq 1$ and

$$f(z) = \begin{cases} 0, & |z| < \epsilon/3 \\ 1, & |z| \geq \epsilon/2. \end{cases} \tag{1.6}$$

Put

$$\widetilde{\tau}_\mu = \begin{cases} \dfrac{f(z)}{z}\dfrac{d}{dz} & \mu = 0 \\ 0 & \mu \geq 1. \end{cases}.$$

Then we have

$$\tau_{\lambda\mu} = \widetilde{\tau}_\mu - \widetilde{\tau}_\lambda$$

and

$$\omega = \overline{\partial}\widetilde{\tau}_\lambda = \frac{1}{z}\left(\frac{\partial f(z)}{\partial\bar{z}}\right)\frac{d}{dz}d\bar{z}$$

represents the Dolbeault cohomology class of $\{\tau_{\lambda\mu}\}$. Since the Dolbeault cohomology class has a representative of the form

$$a\frac{d}{dz}d\bar{z}, \quad a \in \mathbb{C},$$

we will find the constant a for ω defined above. Note that for any C^∞ function h on E we have

$$\int_E (\omega + \overline{\partial}h) \wedge dz = \int_E \omega \wedge dz.$$

We have

$$\begin{aligned}\int_{E-\{|z|<\epsilon\}} \frac{\partial f(z)}{\partial \bar{z}} \frac{1}{z} d\bar{z} \wedge dz &= \int_{E-\{|z|<\epsilon\}} \overline{\partial}(\frac{f(z)}{z} dz) \\ &= \int_{E-\{|z|<\epsilon\}} d(\frac{f(z)}{z} dz) \\ &= -\int_{|z|=\epsilon} \frac{1}{z} dz = -2\pi\sqrt{-1}\end{aligned}$$

by (1.6). On the other hand, we have

$$\int_E a d\bar{z} \wedge dz = 2\sqrt{-1} a \operatorname{Im} \tau.$$

Therefore we conclude that $\{\tau_{\lambda\mu}\}$ and $-\dfrac{\pi}{\operatorname{Im}\tau}\dfrac{d}{dz}d\bar{z}$ represent the same cohomology class. Thus we obtain the equality

$$\theta(z^{-1}\frac{d}{dz}) = -2\pi\sqrt{-1}\rho(\frac{\partial}{\partial\tau}).$$

Next let us consider another coordinate. Put

$$w = \exp(2\pi\sqrt{-1}z), \quad q_0 = \exp(2\pi\sqrt{-1}\tau_0).$$

Then the elliptic curve E can be regarded as the quotient manifold $\mathbb{C}^*/\langle q_0\rangle$, where $\langle q_0\rangle$ means the infinite cyclic group acting on $\mathbb{C}^*$ generated by the analytic automorphism

$$w \mapsto q_0 w.$$

In this coordinate, the origin of the elliptic curve corresponds to the point $w = 1$. Put

$$u = w - 1.$$

Then we have the exact sequence

$$0 \to \Theta_E \to \Theta_E(*o) \xrightarrow{\eta} u^{-1}\mathbb{C}[u^{-1}]\frac{d}{du} \to 0$$

and from this exact sequence we have the exact sequence

$$0 \to H^0(E, \Theta_E(*o)) \xrightarrow{t} u^{-1}\mathbb{C}[u^{-1}]\frac{d}{du} \xrightarrow{\theta} H^1(E, \Theta_E) \to 0.$$

The image $\theta(u^{-1}\dfrac{d}{du})$ is calculated as follows. The image is defined by the cocycle

$$\omega_{\lambda\mu} = \begin{cases} u^{-1}\dfrac{d}{du} & \text{if } \lambda = 0,\ \mu \geq 1 \quad \mathcal{U}_{0\mu} \neq 0 \\ -u^{-1}\dfrac{d}{du} & \text{if } \lambda \geq 1,\ \mu = 0 \quad \mathcal{U}_{\lambda 0} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Since we have

$$\begin{aligned} u^{-1}\frac{d}{du} &= \frac{1}{(\exp(2\pi\sqrt{-1}z) - 1)2\pi\sqrt{-1}\exp(2\pi\sqrt{-1}z)}\frac{d}{dz} \\ &= (\frac{-1}{4\pi^2 z} + \text{holomorphic})\frac{d}{dz} \end{aligned}$$

we conclude

$$\{\omega_{\lambda\mu}\} = -\frac{1}{4\pi^2}\{\tau_{\lambda\tau}\},$$

that is,

$$\theta(u^{-1}\frac{d}{dz}) = -\frac{1}{4\pi^2}\theta(z^{-1}\frac{d}{dz}) = -\frac{1}{2\pi\sqrt{-1}}\rho(\frac{d}{d\tau}).$$

In the following we need to consider degeneration of families of elliptic curves. In that case we shall use the parameter

$$q = \exp(2\pi\sqrt{-1}\tau).$$

Then we have

$$\begin{aligned} q\frac{d}{dq} &= \frac{1}{2\pi\sqrt{-1}}\frac{d}{d\tau} \\ \theta(u^{-1}\frac{d}{du}) &= -\theta(q\frac{d}{dq}). \end{aligned}$$

This formula will be used later.

1.1.4 Kuranishi family. Let us introduce the following definition, to help us understand another aspect of the Kodaira-Spencer mapping.

Definition 1.5 If a complex analytic family $\pi\colon \mathcal{C} \to \mathcal{W}$ of compact Riemann surfaces and the point $0 \in \mathcal{W}$, $C = \pi^{-1}(0)$, satisfy the following condition $(*)$, the family $\pi\colon \mathcal{C} \to \mathcal{W}$ is said to be *complete* at the point 0.

$(*)$ If a complex analytic family $\varpi\colon \mathcal{M} \to \mathcal{N}$ of compact Riemann surfaces, a point $s \in \mathcal{N}$ and an analytic isomorphism $f_0\colon M = \varpi^{-1}(s) \to C$ are given, then there exist a holomorphic mapping g from a neighbourhood U of the point s to $\mathcal{W}$ and a holomorphic mapping $f\colon \varpi^{-1}(U) \to \mathcal{C}$ such that they satisfy the following conditions:

$$\begin{aligned} g(s) &= 0, \\ f|\varpi^{-1}(s) &= f_0, \end{aligned}$$

and such that, for any point $u \in U$, $f|\varpi^{-1}(u)$ is an analytic isomorphism from $\varpi^{-1}(u)$ to $\pi^{-1}\big(g(u)\big)$ with commutative diagram

$$\begin{array}{ccc} \varpi^{-1}(U) & \xrightarrow{f} & \mathcal{C} \\ \downarrow & & \downarrow \\ U & \xrightarrow{g} & \mathcal{W} \end{array} \tag{1.7}$$

In this case the family $\varpi^{-1}(U) \to U$ is complex analytically isomorphic to the pullback of the family $\pi\colon \mathcal{C} \to \mathcal{W}$ by the holomorphic mapping g. Moreover, for all g satisfying the above condition, if $(dg)_s$ is uniquely determined, the family $\pi\colon \mathcal{C} \to \mathcal{W}$ is said to be *versal* at the point 0. Moreover, if g is uniquely determined, the family $\pi\colon \mathcal{C} \to \mathcal{W}$ is said to be *universal* at 0.

If an analytic family $\pi\colon \mathcal{C} \to \mathcal{W}$ is semi-universal but not universal at the point 0, then the analytic automorphism group of the compact Riemann surface $C = \pi^{-1}(0)$ gives an obstruction to being universal. The following theorem is important for applications.

Theorem 1.6 *If a complex analytic family $\pi\colon \mathcal{C} \to \mathcal{W}$ of compact Riemann surfaces of genus $g \geq 2$ is complete at every point of a neighbourhood of a point $0 \in \mathcal{W}$ and versal at 0, then the family π is universal at the point 0.*

The relationship between the Kodaira-Spencer mapping and a complete family or versal family is given in the following theorems.

Theorem 1.7 *If the Kodaira-Spencer mapping of a complex analytic family* $\pi\colon \mathcal{C} \to \mathcal{W}$ *is surjective at a point* $w \in \mathcal{W}$*, then the family is complete at the point* w*.*

Theorem 1.8 *A complex analytic family* $\pi\colon \mathcal{C} \to \mathcal{W}$ *is versal at a point* $0 \in \mathcal{W}$ *if and only if the family* π *is complete at every point of a neighbourhood of the point* 0*, and the Kodaira-Spencer mapping* $\rho_0\colon T_0\mathcal{W} \to H^1\left(\pi^{-1}(0), \Theta\right)$ *at the point* 0 *is an isomorphism.*

1.1.5 Period mappings. Let R be a compact Riemann surface of genus g and $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ be a symplectic basis of $H_1(R, \mathbb{Z})$. A period matrix τ of the Riemann surface R is given by

$$\tau := \left(\int_{\beta_i} \omega_j\right) \cdot \left(\int_{\alpha_i} \omega_j\right)^{-1} \tag{1.8}$$

where $\{\omega_1, \ldots, \omega_g\}$ is a basis of holomorphic one-forms on R. Note that the period matrix τ is independent of the choice of the basis of holomorphic one-forms but depends on the choice of symplectic basis. Let $\{\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_g, \widetilde{\beta}_1, \ldots, \widetilde{\beta}_g\}$ be another symplectic basis of $H_1(R, \mathbb{Z})$. Then we have

$$\begin{pmatrix} \widetilde{\beta_1} \\ \vdots \\ \widetilde{\beta_g} \\ \widetilde{\alpha_1} \\ \vdots \\ \widetilde{\alpha_g} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_g \\ \alpha_1 \\ \vdots \\ \alpha_g \end{pmatrix}$$

where the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is in the group $Sp(g, \mathbb{Z})$ of symplectic integral matrices defined by

$$Sp(g, \mathbb{Z}) := \left\{ X \in M_{2g}(\mathbb{Z}) \mid {}^tX \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} X = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \right\}.$$

Let $\widetilde{\tau}$ be the period matrix of the Riemann surface R defined by the symplectic basis $\{\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_g, \widetilde{\beta}_1, \ldots, \widetilde{\beta}_g\}$. Then we have

$$\widetilde{\tau} = (A\tau + B)(C\tau + D)^{-1}.$$

A basis $\{\omega_1, \ldots, \omega_g\}$ of holomorphic one-forms of R is called a *normalized basis* with respect to a symplectic basis $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ of $H_1(R, \mathbb{Z})$, if we have

$$\int_{\alpha_i} \omega_j = \delta_{ij}, \quad \int_{\beta_i} \omega_j = \tau_{ij}, \quad \tau = (\tau_{ij}). \tag{1.9}$$

Let $\pi : \mathcal{R} \to \mathbf{T}$ be the universal family of compact Riemann surfaces of genus g over the Teichmüller space $\mathbf{T}$. For each point $t \in \mathbf{T}$, put $R_t = \pi^{-1}(t)$. There is a canonical isomorphism

$$H^1(R_t, \Theta_{R_t}) \simeq T_t\mathbf{T} \tag{1.10}$$

which is nothing but the one induced by the Kodaira-Spencer mapping. Taking the dual there is a canonical isomorphism

$$T_t\mathbf{T}^* \simeq H^0(R_t, \omega_{R_t}^{\otimes 2}). \tag{1.11}$$

In the following we often identify the two sides of (1.11). Then, since $\mathbf{T}$ is simply connected, the sheaf $R_1\pi_*\mathbb{Z}$ is trivial on $\mathbf{T}$ and we find global sections

$$\{\widehat{\alpha}_1, \dots, \widehat{\alpha}_g, \widehat{\beta}_1, \dots, \widehat{\beta}_g\}$$

of the sheaf $R_1\pi_*\mathbb{Z}$ such that for each point $t \in \mathbf{T}$,

$$\{\widehat{\alpha}_1(t), \dots, \widehat{\alpha}_g(t), \widehat{\beta}_1(t), \dots, \widehat{\beta}_g(t)\}$$

is a symplectic basis of $H_1(R_t, \mathbb{Z})$. The sheaf $\pi_*\omega_{\mathcal{R}/\mathbf{T}}$ is a free $\mathcal{O}_{\mathbf{T}}$-module of rank g. Let $\{\widehat{\omega}_1, \dots, \widehat{\omega}_g\}$ be an $\mathcal{O}_{\mathbf{T}}$-free basis of $\pi_*\omega_{\mathcal{R}/\mathbf{T}}$. Then the matrix

$$\Pi(t) := \left(\int_{\widehat{\beta}_i(t)} \widehat{\omega}_j(t)\right) \cdot \left(\int_{\widehat{\alpha}_i(t)} \widehat{\omega}_j(t)\right)^{-1} \tag{1.12}$$

defines a holomorphic mapping

$$\Pi : \mathbf{T} \to \mathfrak{S}_g \tag{1.13}$$

from the Teichmüller space to the Siegel upper half-plane of degree g. Moreover, there exists a group homomorphism

$$\Phi : \mathrm{Mod}_g \to Sp(g, \mathbb{Z}) \tag{1.14}$$

of the mapping class group Mod_g of genus g to the symplectic group such that

$$\Pi(\gamma t) = (A_\gamma \Pi(t) + B_\gamma)(C_\gamma \Pi(t) + D_\gamma)^{-1}. \tag{1.15}$$

where

$$\Phi(\gamma) = \begin{pmatrix} A_\gamma & B_\gamma \\ B_\gamma & D_\gamma \end{pmatrix}. \tag{1.16}$$

1.1.6 Jacobian variety of a Riemann surface. For a compact Riemann surface R let $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ be a symplectic basis and $\{\omega_1, \dots, \omega_g\}$ be a normalized basis of holomorphic one-forms. In the following

$$(R, \{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\})$$

is called a *weakly marked Riemann surface*. Let $\tau \in \mathfrak{S}_g$ be the corresponding period matrix. We let $J(R)$ be a g-dimensional complex torus with period matrix (I_g, τ). Then, we can define a holomorphic mapping $j : R \to J(R)$ by

$$j(Q) := (\int_{P_0}^{Q} \omega_1, \dots, \int_{P_0}^{Q} \omega_g) \tag{1.17}$$

where the base point $P_0 \in R$ is fixed once and for all. If we change the base point, the mapping j differs only by a translation of the complex torus $J(R)$. The mapping j is called the *Albanese* mapping and the complex torus $J(R)$ is usually called the Jacobian variety of the Riemann surface R.

The Jacobian variety $J(R)$ may be defined intrinsically by

$$J(R) = H^0(R, \omega_R)^*/H_1(R, \mathbb{Z}). \tag{1.18}$$

Precisely speaking, this is the definition of the Albanese variety of the Riemann surface R. Since we have

$$\int_{\alpha_i} \omega_j = \delta_{ij}$$

for a normalized basis $\{\omega_1, \dots, \omega_g\}$ of $H^0(R, \omega_R)$ we may regard $\{\alpha_1, \dots, \alpha_g\}$ as the corresponding dual basis of $H^0(R, \omega_R)^*$. Then the equality

$$\int_{\beta_i} \omega_j = \tau_{ij}$$

implies that

$$\beta_i = \sum_{j=1}^{g} \tau_{ij} \alpha_j.$$

Therefore, we can express $J(R)$ as a complex torus $\mathbb{C}^g/(I_g, \tau)$ defined above.

The Picard variety $Pic^0(R)$ of the Riemann surface R is defined by

$$Pic^0(R) := H^1(R, \mathcal{O}_R)/H^1(R, \mathbb{Z}). \tag{1.19}$$

Choose again a normalized basis $\{\omega_1, \dots, \omega_g\}$ of $H^0(R, \omega_R)$. Then $\{\overline{\omega}_1, \dots, \overline{\omega}_g\}$ is a basis of $H^1(R, \mathcal{O}_R)$. By the Hodge decomposition theorem, we have an isomorphism

$$H^1(R, \mathbb{C}) \simeq H^1(R, \mathcal{O}_R) \oplus H^0(R, \omega_R).$$

Let $\{\alpha_1^*, \dots, \alpha_g^*, \beta_1^*, \dots, \beta_g^*\}$ be the basis of the cohomology group $H^1(R, \mathbb{Z})$ dual to the basis $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ of $H_1(R, \mathbb{Z})$. Since we may regard $H^1(R, \mathbb{Z})$ as a lattice of $H^1(R, \mathbb{C})$, by the above isomorphism we may express

$$\begin{aligned} \alpha_i^* &= \sum_{j=1}^{g} a_{ij} \overline{\omega}_j + \sum_{j=1}^{g} \overline{a}_{ij} \omega_j \\ \beta_i^* &= \sum_{j=1}^{g} b_{ij} \overline{\omega}_j + \sum_{j=1}^{g} \overline{b}_{ij} \omega_j. \end{aligned}$$

Put

$$A = (a_{ij}) \quad B = (b_{ij}).$$

By the equalities

$$\begin{aligned} \delta_{ik} = \langle \alpha_k, \alpha_i^* \rangle &= \sum_{j=1}^{g} a_{ij} \int_{\alpha_k} \overline{\omega}_j + \sum_{j=1}^{g} \int_{\alpha_k} \overline{a}_{ij} \omega_j \\ &= a_{ik} + \overline{a}_{ik} \\ 0 = \langle \beta_k, \alpha_i^* \rangle &= \sum_{j=1}^{g} a_{ij} \int_{\beta_k} \overline{\omega}_j + \sum_{j=1}^{g} \overline{a}_{ij} \int_{\beta_k} \omega_j \\ & \quad \sum_{j=1}^{g} a_{ij} \overline{\tau}_{jk} + \sum_{j=1}^{g} \overline{a}_{ij} \tau_{jk}. \end{aligned}$$

and the similar equalities for β_i^*, we have

$$\begin{pmatrix} A & \overline{A} \\ B & \overline{B} \end{pmatrix} \begin{pmatrix} I_g & \overline{\tau} \\ I_g & \tau \end{pmatrix} = I_{2g}$$

Put

$$Y = (-2\pi\sqrt{-1} \operatorname{Im} \tau)^{-1}. \tag{1.20}$$

Note that Y is a symmetric matrix. We have

$$\begin{pmatrix} I_g & \overline{\tau} \\ I_g & \tau \end{pmatrix}^{-1} = \begin{pmatrix} -\tau Y & -\overline{\tau} Y \\ Y & Y \end{pmatrix}.$$

Hence the lattice induced by $H^1(R,\mathbb{Z})$ in $H^1(R,\mathcal{O}_R)$ is given by the rows of the matrix

$$\begin{pmatrix} -\tau Y \\ Y \end{pmatrix}$$

where we use $\{\overline{\omega}_1,\dots,\overline{\omega}_g\}$ as a basis of $H^1(R,\mathcal{O}_R)$. Hence, we may express $Pic^0(R)$ as a complex torus $\mathbb{C}^g/(Y\tau, Y)$. Thus, there is an isomorphism

$$\lambda : J(R) \to Pic^0(R)$$

given by

$$\lambda \begin{bmatrix} z_1 \\ \vdots \\ z_g \end{bmatrix} = \left[Y \begin{pmatrix} z_1 \\ \vdots \\ z_g \end{pmatrix} \right]. \tag{1.21}$$

Note that the isomorphism λ is induced from a $\mathbb{C}$-linear isomorphism

$$\begin{array}{ccc} H^0(R,\omega_R)^* & \to & H^1(R,\mathcal{O}_R) \\ \sum a_i\alpha_i & \mapsto & \sum_{ij}(y_{ij}a_j)\overline{\omega}_i \end{array} \tag{1.22}$$

where $(y_{ij}) = Y$.

The Albanese and Picard varieties of a compact Riemann surface are dual to each other and they are principally polarized abelian varieties. The above isomorphism $\lambda : J(R) \to Pic^0(R)$ is a canonical one. The sheaf $\Theta_{J(R)}$ of holomorphic vector fields on the Jacobian variety is trivial and we have a canonical isomorphism

$$\Theta_{J(R)} \simeq T_0J(R) \otimes \mathcal{O}_{J(R)}$$

Moreover, the Albanese mapping $j : R \to J(R)$ induces an isomorphism

$$H^1(J(R),\mathcal{O}) \simeq H^1(R,\mathcal{O})$$

by $d\overline{z}_j \mapsto \overline{\omega}_j$. Therefore, we have a canonical isomorphism

$$\iota : H^1(J(R),\Theta_{J(R)}) \simeq T_0J(R) \otimes H^1(R,\mathcal{O}_R).$$

where $T_0J(R)$ is the tangent space of $J(R)$ at the origin. By the above isomorphism λ, we may identify $T_0J(R)$ with $H^1(R,\mathcal{O}_R)$. Hence, we may rewrite

$$\iota : H^1(J(R),\Theta_{J(R)}) \simeq H^1(R,\mathcal{O}_R) \otimes H^1(R,\mathcal{O}_R). \tag{1.23}$$

Since $T_0J(R)$ is identified with $H^0(R,\omega_R)^*$ by (1.19), by (1.8)) the mapping ι is written explicitly as

$$\iota(\frac{\partial}{\partial z_i} d\overline{z}_j) = \sum_{k=1}^{g} y_{ik}\overline{\omega}_k \otimes \overline{\omega}_j \tag{1.24}$$

where the matrix $(y_{ij}) = Y$ is defined in (1.20). Put

$$\mathfrak{T} := \{T = (t_{ij}) | T \text{ is } g\times g \text{ complex matrix with} \det \begin{pmatrix} I_g & T \\ I_g & \overline{T} \end{pmatrix} \neq 0\}$$

and we let $\mathbb{Z}^{2g}$ act on $\mathfrak{T} \times \mathbb{C}^g$ by

$$\underline{n} : (T,z) \to (T, z + \underline{n}_1 + \underline{n}_2 T), \quad \underline{n} = (\underline{n}_1, \underline{n}_2), \quad \underline{n}_i \in \mathbb{Z}^g.$$

The quotient space $\pi : \mathcal{T} = \mathfrak{T} \times \mathbb{C}^g/\mathbb{Z}^{2g} \to \mathfrak{T}$ is a versal family of complex tori. Let $\mu : \mathcal{A} \to \mathfrak{S}_g$ be the family of abelian varieties over the Siegel upper half-plane such that the fibre $A_\tau = \mu^{-1}(\tau)$ is the complex torus with period matrix (I_g, τ). There is a natural embedding $i : \mathfrak{S}_g \hookrightarrow \mathfrak{T}$ and the induced family $i^*\mathcal{T} \to \mathfrak{S}_g$ is isomorphic to the family $\mu : \mathcal{A} \to \mathfrak{S}_g$. We call the mapping

$$i \circ \Pi : \mathbf{T} \to \mathfrak{T}$$

also the period mapping. At a point $\tau \in \mathfrak{S}_g \subset \mathfrak{T}$ corresponding to the Riemann surface R with symplectic basis $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$, by the Kodaira-Spencer mapping we may identify the tangent space $T_\tau\mathfrak{T}$ of $\mathfrak{T}$ at τ with $H^1(J(R), \Theta_{J(R)})$. Hence, the period mapping $i \circ \Pi$ induces a natural mapping

$$H^1(R, \Theta_R) \to H^1(J(R), \Theta_{J(R)}) = H^1(R, \mathcal{O}_R) \otimes H^1(R, \mathcal{O}_R)$$

where we use the isomorphism ι (1.22). The dual of this mapping is given by the natural multiplication map

$$\begin{array}{ccc} H^0(R, \omega_R) \otimes H^0(R, \omega_R) & \to & H^0(R, \omega_R^{\otimes 2}) \\ \nu \otimes \mu & \mapsto & \nu\mu. \end{array} \tag{1.25}$$

Let us calculate the Kodaira-Spencer class of a tangent vector $\dfrac{\partial}{\partial t_{ij}}$ at a point $\tau \in \mathfrak{S}_g \subset \mathfrak{T}$. Let $\mathcal{U}$ be an open neighbourhood of the point τ_0 and let $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ be a small open covering of $\mathcal{U}$. We choose local coordinates $(t_{ij}, z_\lambda^1, \ldots, z_\lambda^g)$, $1 \le i \le j \le g$ in such a way that $(z_\lambda^1, \ldots, z_\lambda^g)$ are deduced from the global affine coordinates $(z_1, \ldots, z_g)$ of the vector space $\mathbb{C}^g$. Thus, if $\mathcal{U}_\lambda \cap \mathcal{U}_\mu \neq \emptyset$, then we have

$$z_\lambda^i - z_\mu^i = m_{\lambda\mu}^i + \sum_{k=1}^g n_{\lambda\mu}^k t_{ik} \tag{1.26}$$

where $m_{\lambda\mu}^i$, $n_{\lambda\mu}^i \in \mathbb{R}^g$, $i = 1, 2, \ldots, g$. Therefore, the Kodaira-Spencer class $\rho(\dfrac{\partial}{\partial t_{ij}})$ is given by the cocycle $\theta_{\lambda\mu}$ defined by

$$\theta_{\lambda\mu} = n_{\lambda\mu}^j \frac{\partial}{\partial z_\lambda^i}. \tag{1.27}$$

Now put

$$\begin{pmatrix} \psi_\lambda^1 \\ \vdots \\ \psi_\lambda^g \end{pmatrix} = Y \begin{pmatrix} \overline{z}_\lambda^1 - z_\lambda^1 \\ \vdots \\ \overline{z}_\lambda^g - z_\lambda^g \end{pmatrix}.$$

where $Y = (-2\sqrt{-1}\mathrm{Im}(t_{ij}))^{-1}$. Then we have

$$\psi_\mu^j \frac{\partial}{\partial z_\mu^i} - \psi_\lambda^j \frac{\partial}{\partial z_\lambda^i} = \theta_{\lambda\mu}.$$

Thus

$$\overline{\partial}(\psi_\mu^j \frac{\partial}{\partial z_\mu^i}) = \overline{\partial}(\psi_\lambda^j \frac{\partial}{\partial z_\lambda^i})$$

is a global form and is expressed as

$$\sum_{k=1}^g y_{jk} \frac{\partial}{\partial z_i} d\overline{z}_k. \tag{1.28}$$

The Dolbeault cohomology class (1.0-20) is the Kodaira-Spencer class $\rho(\frac{\partial}{\partial t_{ij}})$ and by (1.22) we have

$$\iota(\rho(\frac{\partial}{\partial t_{ij}})) = \sum_{\ell=1}^{g}\sum_{k=1}^{g}\{(y_{i\ell}\overline{\omega}_\ell)\otimes(y_{jk}\overline{\omega}_k)\}$$

Hence, by (1.22) the dual of this class in $H^0(R,\omega_R)\otimes H^0(R,\omega_R)$ is $\omega_i\otimes\omega_j$. Hence, by (1.26) we conclude

$$\Pi^*(d\tau_{ij}) = \Pi^*\circ i^*(dt_{ij}) = \omega_i\omega_j \tag{1.29}$$

where we use the identification (1.11). Thus we have proved the following Proposition.

Proposition 1.9 *Under the above assumption and notation, we have*

$$\Pi^*(\sum_{i<j}\frac{\partial}{\partial\tau_{ij}}(\log\det(C_\gamma\tau + D_\gamma))d\tau_{ij}) \tag{1.30}$$
$$= \sum_{i<j}\omega_i\omega_j\Pi^*(\frac{\partial}{\partial\tau_{ij}}\log\det(C_\gamma\tau + D_\gamma)).$$

1.1.7 Symmetric bidifferential. Let us consider a symmetric bidifferential form on a compact Riemann surface. First we recall basic facts on theta functions. For a point $\tau\in\mathfrak{S}_g$ the theta function $\theta(\tau,z)$ is defined as

$$\theta(\tau,z) := \sum_{m\in\mathbf{Z}^g}\exp\{2\pi\sqrt{-1}(\frac{1}{2}{}^t m\tau m + {}^t m z)\}.$$

The theta function $\theta(\tau,z)$ can be considered as a holomorphic section of the line bundle which defines a principal polarization of the abelian variety $A_\tau = \mathbb{C}^g/(I_g,\tau)$. For $e\in\mathbb{C}^g$ we denote the corresponding point of A_τ by $[e]$. The zeroes of the theta function $\theta(\tau,z)$ define a divisor on A_τ called the *theta divisor* and is denoted by Θ.

The theta function $\theta[e](\tau,z)$ with characteristic $e = \begin{bmatrix}\delta\\ \epsilon\end{bmatrix}$, $\delta,\epsilon\in\mathbb{R}^g$ is defined as

$$\theta[e](\tau,z) := \sum_{m\in\mathbb{Z}^g}\exp 2\pi\sqrt{-1}\{\frac{1}{2}{}^t(m+\delta)\tau(m+\delta) + {}^t(m+\delta)(z+\epsilon))\}.$$

Let R be a compact Riemann surface of genus g. Let us fix a symplectic basis of $H_1(R,\mathbb{Z})$ and let $\tau\in\mathfrak{S}_g$ be the corresponding period matrix. Then the abelian variety A_τ is the Jacobian variety $J(R)$ of R. Let $Div^d(R)$ be the group of divisors of degree d on a compact Riemann surface R. We can extend the mapping $j: R\to J(R)$ to the mapping $j: Div^d(R)\to J(R)$ by putting

$$j(\sum_{i=1}^{m+d}P_i - \sum_{k=1}^{m}Q_k) := \sum_{i=1}^{m+d}j(P_i) - \sum_{k=1}^{m}j(Q_k).$$

Note that for two divisors $D, D'\in Div^d(R)$, $j(D) = j(D')$ if and only if D and D' are linearly equivalent, that is, there is a meromorphic function f on R such that $D = D' + (f)$. Let $Pic^d(R)$ be the set of isomorphism classes of line bundles of degree d on the compact Riemann surface R. Then, by Abel's theorem, the mapping $j: Div^d(R)\to J(R)$ factors through $j: Pic^d(R)\to J(R)$, where we identify $Pic^d(R)$ with the quotient $Div^d(R)/\sim$ by the linear equivalence relation. Moreover, for any divisor D on R we denote the corresponding line bundle on R by $[D]$. Also in the following we always identify $Pic^0(R)$ with $J(R)$ via the canonical isomorphism λ. The following Theorem due to Riemann is the most fundamental in the theory of theta functions associated with a Riemann surface.

Theorem 1.10 *There is a divisor* $\Delta \in Div^{g-1}(R)$ *with*

$$[2\Delta] = \omega_R \tag{1.31}$$

such that for any point $P \in R$ *and* $e \in \mathbb{C}^g$ *we have:*

(1) *If* $\theta(\tau, e) \neq 0$*, then the divisor* D *of* R *defined by the equation in the variable* Q

$$\theta(\tau, j(Q) - j(P) - e) = 0$$

is an effective divisor of degree g *with*

$$H^1(R, \mathcal{O}_R(D)) = 0$$

and

$$[e] = j(D - P - \Delta).$$

(2) *If* $\theta(\tau, e) = 0$*, then for some effective divisor* E *of degree* $g-1$ *we have*

$$[e] = j(E - \Delta).$$

Moreover, $\dim_{\mathbb{C}} H^1(R, \mathcal{O}_R(E))$ *is the multiplicity of the theta divisor* Θ *at the point* $[e]$ *and is the smallest integer* d *such that*

$$\theta(B - A - e) \equiv 0$$

for all effective divisors A*,* B *of degree less than or equal to* $d-1$*.*

Put

$$L_0 = [\Delta]. \tag{1.32}$$

Hence, we have

$$L_0^{\otimes 2} = \omega_R. \tag{1.33}$$

For a half-period (i.e. a two-torsion point) $\alpha \in J(C)$, that is, $2\alpha = 0$ in $J(R)$, we let L_α be the line bundle on the Riemann surface R such that $L_\alpha \otimes L_0^{-1}$ is the half period α. Hence, in particular

$$L_\alpha^{\otimes 2} = \omega_R. \tag{1.34}$$

Corollary 1.11 *We have*

$$h^0(R, L_\alpha) = h^1(R, L_\alpha) = \mathrm{mult}_\alpha \Theta \tag{1.35}$$

and $h^0(R, L_\alpha)$ *is even for* α *even and odd for* α *odd, where a half-period* α *is called odd (resp. even), if we have*

$$\alpha = \frac{1}{2} \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} \begin{pmatrix} \tau \\ I_g \end{pmatrix}, \quad \delta,\, \epsilon \in \mathbb{Z}^g$$

with $\delta \cdot {}^t\epsilon$ *even (resp. odd).*

For the proofs of Theorem 1.10 and Corollary 1.11 we refer the reader to [Fay]. Let α be a non-singular odd half-period, that is, $\mathrm{mult}_\alpha \Theta = 1$. Then, by the above corollary, the line bundle L_α has a non-zero global section h_α satisfying

$$h_\alpha^2 = \sum_{i=1}^{g} \frac{\partial \theta[\alpha]}{\partial z_i}(\tau, 0)\omega_i \tag{1.36}$$

where $\{\omega_1, \ldots, \omega_g\}$ is the normalized basis of holomorphic one-forms and we use (1.34). The *prime form* $E(P, Q)$ of the Riemann surface R is defined as

$$E(P,Q) := \frac{\theta[\alpha](\tau, j(P-Q))}{h_\alpha(P)h_\alpha(Q)} \tag{1.37}$$

for points P, $Q \in R$. Note that the prime form is independent of the choice of α and is a holomorphic section of the line bundle $\pi_1 L_0^{-1} \otimes \pi_2 L_0^{-1} \otimes \delta^*(\Theta)$ on $R \times R$, where π_i is the projection of $R \times R$ to the i-th factor and $\delta : R \times R \to J(R)$ is the mapping sending $(P, Q) \in R \times R$ to $j(Q - P)$. Moreover, we have

$$E(P,Q) = -E(Q,P) \tag{1.38}$$

and $E(P, Q)$ vanishes to first order along the diagonal in $R \times R$ and does not vanish outside the diagonal. Let x (resp. y) be a local coordinate of R with center P (resp. Q) and by abuse of notation let us write

$$\begin{aligned} h_\alpha(P) &= h_\alpha(x)\sqrt{dx} \\ h_\alpha(Q) &= h_\alpha(y)\sqrt{dy}. \end{aligned}$$

Then the prime form $E(P, Q)$ is expressed as

$$E(P,Q) = E(x,y)(\sqrt{dy})^{-1}(\sqrt{dy})^{-1}.$$

Now let us define the bidifferential $\omega(x, y)dxdy$ by

$$\omega(x,y)dxdy := \frac{\partial^2}{\partial x \partial y} \log E(x,y)dxdy. \tag{1.39}$$

Then $\omega(x, y)dxdy$ is a meromorphic bidifferential on $R \times R$ holomorphic outside the diagonal and has a pole of order two along the diagonal. That is, $\omega(x, y)dxdy \in H^0(R \times R, p_1^*\omega_R \otimes p_2^*\omega_R(2\Delta))$ Moreover, by (1.38) we have

$$\omega(x,y) = \omega(y,x). \tag{1.40}$$

Proposition 1.12 *For $f \in \mathbb{C}^g$, if $[f] \in J(R)$ is a non-singular point of the theta divisor Θ, then we have*

$$\omega(x,y)dxdy = \frac{\partial^2 \log\theta}{\partial x \partial y}(\tau, j(x) - j(y) - f)dxdy.$$

For a proof, see [Fay].

For a symmetric bidifferential $\widehat{\omega}(x, y)dxdy$ on $R \times R$ holomorphic outside the diagonal, we always have an expansion along the diagonal

$$\widehat{\omega}(x,y)dxdy = \left(\sum_{k=k_0}^{2} \frac{c_{-k}}{(x-y)^k} + \text{holomorphic} \right).$$

The coefficient c_{-2} is independent of the choice of local coordinates and called the *biresidue* and written as $\mathrm{Res}^2(\widehat{\omega})$.

Corollary 1.13 *There exists a symmetric bidifferential ω which is holomorphic outside the diagonal and has a pole of order 2 on the diagonal with* $\mathrm{Res}^2\,\omega = 1$.

In the following, by a meromorphic symmetric bidifferential $\omega(x, y)dxdy$ of a Riemann surface R we mean that $\omega(y, x) = \omega(x, y)$, ω is holomorphic on $R \times R$ outside the diagonal and has a pole of order 2 along the diagonal.

Lemma 1.14 *Let $\omega(x,y)dxdy$ and $\widetilde{\omega}(x,y)dxdy$ be symmetric bilinear differentials on $R \times R$ holomorphic outside the diagonal such that they have poles of order 2 on the diagonal with*

$$\mathrm{Res}^2(\omega) = \mathrm{Res}^2(\widetilde{\omega})$$

Then the difference

$$\Omega(x,y)dxdy = \omega(x,y)dxdy - \widetilde{\omega}(x,y)dxdy$$

is a holomorphic symmetric bidifferential on $R \times R$.

The space of symmetric bilinear differentials on $R \times R$ having a pole of order two on the diagonal and holomorphic outside the diagonal is written as

$$H^0(R \times R, p_1^*\omega_R \otimes p_2^*\omega_R(2\Delta))^{\langle\iota\rangle}$$

where ι is the automorphism of $R \times R$ exchanging the components. Note that

$$\omega_{R\times R} = p_1^*\omega_R \otimes p_2^*\omega_R$$

Let us define the *projective connection* $S_\omega(z)dz^2$ attached to a symmetric bidifferential ω with biresidue 1 of a compact Riemann surface R by

$$S_\omega(z)dz^2 = 6\lim_{w\to z}(\omega(w,z)dwdz - \frac{dwdw}{(w-z)^2}). \tag{1.41}$$

The projective connection $S_\omega(z)dz^2$ does depend on a choice of a local coordinate z. The following theorem is well-known and is obtained by direct calculations.

Theorem 1.15

$$S_\omega(z)dz^2 = S_\omega(w)dw^2 - \{z;w\}dw^2$$

where $\{z;w\}$ is the Schwarzian derivative defined by

$$\{z;w\} = \frac{d^3z}{dw^3}/\frac{dz}{dw} - \frac{3}{2}\left(\frac{d^2z}{dw^2}/\frac{dz}{dw}\right)^2.$$

1.2 Stable N-pointed curves

1.2.1 Nodal curve. By a nodal curve C we mean a closed analytic subvariety in a complex projective space $\mathbb{P}^n$ whose singular points are nodes, that is, local analytically isomorphic to a neighbourhood $U_\epsilon = \{(z,w) \mid zw = 0,\ |z| < \epsilon, |w| < \epsilon\}$ of $zw = 0$ at the origin. If we use the language of algebraic geometry, a nodal curve is a semi-stable curve, that is, a reduced connected complete algebraic curve defined over the complex numbers $\mathbb{C}$. let P be a node of a nodal curve C. There is a neighbourhood of P isomorphic to the above U_ϵ for sufficiently small ϵ. Then we can desingularize the node P by separating $U_+ = U_\epsilon \cap \{w = 0\} = \{\, z \mid |z| < \epsilon\}$ and $U_- = U_\epsilon \cap \{z = 0\} = \{\, w \mid |w| < \epsilon\}$. Hence, instead of the point P we have two points $P_+ = 0 \in U_+$ and $P_- = 0 \in U_-$. The new variety $\widetilde{C}$ thus obtained is called the curve obtained by normalization at the point P. There is a natural holomorphic mapping $\varpi : \widetilde{C} \to C$ such that ϖ is the identity mapping from $\widetilde{C} \setminus \{P_+, P_-\}$ to $C \setminus \{P\}$ and $\varpi^{-1}(P) = \{P_+, P_-\}$.

Conversely, from a compact Riemann surface we can construct a nodal curve. Let P_+ and P_- be two distinct points on a compact Riemann surface. Choose a local coordinate z (resp. w) of R with center P_+ (resp. P_-). Let us identify P_+ with P_- and obtain a new space C. By P we denote the point of C corresponding to $P_\pm$. We identify a neighbourhood of P in C with a neighbourhood of $zw = 0$ at

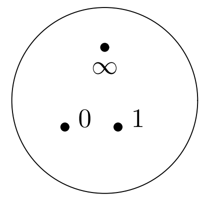

Figure 1.2 Stable 3-pointed curves of genus 0.

the origin. Identify $C \setminus \{P\}$ with $R \setminus \{P_+, P_-\}$. Then C is a nodal curve. If we choose two other points on $C \setminus \{P\}$ and repeat the same process we obtain a nodal curve with two nodes. In this way any nodal curve is obtained from a compact Riemann surface.

1.2.2 Stable N-pointed curves.

Definition 1.16 The data $\mathfrak{X} = (C; Q_1, Q_2, \ldots, Q_N)$ consisting of a curve C and points $Q_1, \ldots, Q_N$ on C is called an *stable N-pointed curve* if the following conditions are satisfied.

(1) The curve C is a compact Riemann surface or a nodal curve.
(2) $Q_1, Q_2, \ldots, Q_N$ are non-singular points of the curve C.
(3) If an irreducible component C_i is a projective line (i.e. Riemann sphere) $\mathbb{P}^1$ (resp. a rational curve with one double point, resp. an elliptic curve), the sum of the number of intersection points of C_i and other components and the number of Q_j's on C_i is at least three (resp. one).
(4) $\dim_{\mathbb{C}} H^1(C, \mathcal{O}_C) = g$.

Note that the above condition (3) is equivalent to saying that $\mathrm{Aut}(\mathfrak{X})$ is a finite group, so that $\mathfrak{X}$ has no infinitesimal automorphisms. In the following we often add the following condition (Q) for an N-pointed stable curve $\mathfrak{X}$.

(Q) Each component C_i contains at least one Q_j.

The meaning of the condition (Q) will be clarified in the following Lemma 1.21 and Lemma 1.22.

Example 1.17 For g =0, by the property (3), $N \geq 3$. Any 3-pointed stable curve of genus 0 is isomorphic to the curve in Figure 1.2.

Any 4-pointed stable curve of genus 0 is isomorphic to one of the following two stable curves in Figure 1.3.

Any 1-pointed stable curve of genus 1 is isomorphic to one of the following two stable curves in Figure 1.4.

Definition 1.18 Let C be a curve and Q a non-singular point on C. The *n-th infinitesimal neighbourhood* $s^{(n)}$ of C at the point Q is the $\mathbb{C}$-algebra isomorphism

$$s^{(n)} : \mathcal{O}_{C,Q}/\mathfrak{m}_Q^{n+1} \simeq \mathbb{C}[[\xi]]/(\xi^{n+1}) \tag{1.42}$$

where $\mathfrak{m}_Q$ is the maximal ideal of $\mathcal{O}_{C,Q}$ consisting of germs of holomorphic functions vanishing at Q.

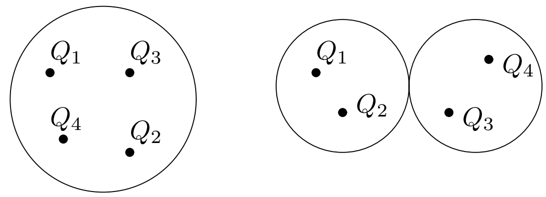

Figure 1.3 Stable 4-pointed curves of genus 0.

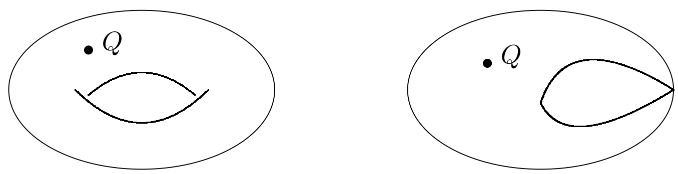

Figure 1.4 Stable 1-pointed curves of genus 1.

Taking the limit $n \to \infty$ in the isomorphism (1.42), we have the isomorphism

$$s^{(\infty)} : \widehat{\mathcal{O}}_{C,Q} \simeq \mathbb{C}[[\,\xi\,]]. \tag{1.43}$$

The isomorphism $s^{(\infty)}$ is called the *formal neighbourhood* of C at Q.

Definition 1.19 The data $\mathfrak{X} = (C; Q_1, Q_2, \ldots, Q_N\,; s_1, s_2, \ldots, s_N)$ is called an *stable N-pointed curve of genus g with formal neighbourhoods* if

(1) $(C\,; Q_1, Q_2, \ldots, Q_N)$ is an N-pointed stable curve of genus g.
(2) s_j is a formal neighbourhood of C at Q_j.

Similarly an stable N-pointed curve with n-th formal neighbourhoods

$$\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \ldots, Q_N\,; s_1^{(n))}, s_2^{(n))}, \ldots, s_N^{(n))})$$

is defined .

1.2.3 Residue pairing. Let C be a nodal curve, that is, C is a reduced curve with at most ordinary double points and proper over $\mathbb{C}$. Let Ω^1_C be the sheaf of Kähler differentials of the curve C and ω_C be the dualizing sheaf of the curve C. Near a singular point P, the curve C is analytically isomorphic to the variety defined by

$$xy = 0.$$

In these coordinates the sheaf Ω^1_C is expressed as

$$\Omega^1_C = (\mathcal{O}_C dx + \mathcal{O}_C dy)/(xdy + ydx)\mathcal{O}_C. \tag{1.44}$$

On the other hand, near the singular point P the dualizing sheaf ω_C is the invertible sheaf generated by the differential ζ given by dx/x outside $x = 0$ and $-dy/y$ outside $y = 0$. Moreover, outside the singular points of the curve C, the sheaves Ω^1_C and ω_C coincide. Thus, we have

$$\Omega^1_C = \mathfrak{m}\omega_C \tag{1.45}$$

where $\mathfrak{m}$ is the defining ideal sheaf of the singular points of C. Hence, we have the following exact sequence.

$$0 \to \Omega^1_C \to \omega_C \to \omega_C \otimes \mathcal{O}_{C_{Sing}} \to 0.$$

Let $\nu : \widetilde{C} \to C$ be the normalization of the curve C. We let $\{P_1, \dots, P_q\}$ be the set of double points of the curve C and for each double point P_i, put $\nu^{-1}(P_i) = \{ P_{i,+}, P_{i,-} \}$. Then, we have the following exact sequence.

$$0 \to \omega_C \to \nu_* \omega_{\widetilde{C}}(\sum_i^k (P_{i,+} + P_{i,-})) \xrightarrow{r} \bigoplus_i^q \mathbb{C} \to 0 \tag{1.46}$$

where at each double point P_i, the mapping r is given by

$$\mathrm{res}_{P_{i,+}}(\tau) - \mathrm{res}_{P_{i,-}}(\tau).$$

This means that a local holomorphic section of the dualizing sheaf ω_C is regarded as a local meromorphic section of a one-form on $\widetilde{C}$ which has a pole of order one at $P_{i,+}$ and $P_{i,-}$ such that the sum of the residues is zero and the one-form is holomorphic outside $P_{i,\pm}$'s. In the following we shall often use this interpretation. The following Lemma is an easy consequence of this interpretation.

Lemma 1.20 *Let τ be a meromorphic section of the dualizing sheaf ω_C holomorphic at the double points. Then the sum of the residues of τ is zero.*

Lemma 1.21 *Assume that the N-pointed stable curve*

$$\mathfrak{X} = (C; Q_1, Q_2, \dots, Q_N \,;\, s_1, s_2, \dots, s_N)$$

with formal neighbourhoods satisfies the condition (Q). By t_j we denote the Laurent expansions at Q_j with respect to a formal parameter $\xi_j = s^{-1}(\xi)$. Then the following homomorphisms are injective:

$$t = \oplus t_j \quad : \quad H^0(C, \mathcal{O}(* \sum_{j=1}^N Q_j)) \longrightarrow \bigoplus_{j=1}^N \mathbb{C}((\xi_j)) \tag{1.47}$$

$$t = \oplus t_j \quad : \quad H^0(C, \omega_C(* \sum_{j=1}^N Q_j)) \longrightarrow \bigoplus_{j=1}^N \mathbb{C}((\xi_j)) d\xi_j \tag{1.48}$$

where ω_C is the dualizing sheaf of the curve C and

$$H^0(C, \mathcal{O}(* \sum_{j=1}^N Q_j)) = \operatorname*{inj\,lim}_{n \to \infty} H^0(C, \mathcal{O}(n \sum_{j=1}^N Q_j)).$$

Proof By condition (Q) if an element $f \in H^0(C, \mathcal{O}(* \sum_{j=1}^N Q_j))$ is zero at an irreducible component of the curve C, then $f = 0$. Hence the mapping (1.47) is injective. Similarly, if an element $\omega \in H^0(C, \omega_C(* \sum_{j=1}^N Q_j))$ is zero at an irreducible component of the curve C, then $\omega = 0$, and hence the injectivity of (1.48) follows. $\square$

By this Lemma $H^0(C, \mathcal{O}(* \sum_{j=1}^N Q_j))$ (resp. $H^0(C, \omega_C(* \sum_{j=1}^N Q_j)))$ can be regarded as a subspace of $\oplus_{j=1}^N \mathbb{C}((\xi_j))$ (resp. $\oplus_{j=1}^N \mathbb{C}((\xi_j)) d\xi_j$). There is the residue pairing

$$\begin{array}{ccc}\bigoplus_{j=1}^{N}\mathbb{C}((\xi_j))\times\bigoplus_{j=1}^{N}\mathbb{C}((\xi_j))d\xi_j & \to & \mathbb{C} \\ ((f(\xi_1),\dots,f(\xi_N)),g(\xi_1)d\xi_1,\dots,g(\xi_N)d\xi_N) & \mapsto & \sum_{j=1}^{N}\operatorname*{Res}_{\xi_j=0}(f(\xi_j)g(\xi_j)d\xi_j).\end{array} \tag{1.49}$$

The following Theorem plays an important role in our theory.

Theorem 1.22 *Under the residue pairing the vector space* $H^0(C,\mathcal{O}(*\sum_{j=1}^N Q_j))$ *and the vector space* $H^0(C,\omega_C(*\sum_{j=1}^N Q_j))$ *are annihilators of each other.*

Proof Since the residue is independent of the choice of local coordinates, we may assume that the formal neighbourhood gives holomorphic coordinates in a neighbourhood of the point Q_j. For any positive integers m, n we have the exact sequence

$$0\to\mathcal{O}_{\mathcal{C}}(-m\sum_{j=1}^{N}Q_j))\to\mathcal{O}_{\mathcal{C}}(n\sum_{j=1}^{N}Q_j)\xrightarrow{p}\bigoplus_{j=1}^{N}\bigoplus_{k=-n}^{m-1}\mathbb{C}\xi_j^k\to 0. \tag{1.50}$$

From this exact sequence we have the long exact sequence of cohomology groups

$$H^0(C,\mathcal{O}_{\mathcal{C}}(n\sum_{j=1}^{N}Q_j))\to\bigoplus_{j=1}^{N}\bigoplus_{k=-n}^{m-1}\mathbb{C}\xi_j^k\xrightarrow{c}H^1(C,\mathcal{O}_{\mathcal{C}}(-m\sum_{j=1}^{N}Q_j))\to 0.$$

Note that by the Serre duality $H^1(C,\mathcal{O}_{\mathcal{C}}(-m\sum_{j=1}^N Q_j))$ is dual to

$$H^0(C,\omega_C(m\sum_{j=1}^{N}Q_j)).$$

On the image of c the dual pairing

$$\langle\ ,\ \rangle : H^1(C,\mathcal{O}_{\mathcal{C}}(-m\sum_{j=1}^{N}Q_j))\times H^0(C,\mathcal{O}_{\mathcal{C}}(m\sum_{j=1}^{N}Q_j))\to\mathbb{C}$$

is given by

$$\langle c(((g_j(\xi_j))),\tau\rangle=\sum_{j=1}^{N}\operatorname*{Res}_{\xi_j=0}(g_j(\xi_j)\tau_j) \tag{1.51}$$

where $(g_j(\xi_j))\in\bigoplus_{j=1}^N\bigoplus_{k=-n}^{m-1}\mathbb{C}\xi_j^k$ $\tau\in H^0(C,\mathcal{O}_{\mathcal{C}}(m\sum_{j=1}^N Q_j))$ and τ_j is the Laurent expansion of τ at the point Q_j with respect to ξ_j. $(g_j(\xi_j))\in\bigoplus_{j=1}^N\bigoplus_{k=-n}^{m-1}\mathbb{C}\xi_j^k$ is the image of a global meromorphic function $g\in H^0(C,\mathcal{O}_{\mathcal{C}}(-m\sum_{j=1}^N Q_j))$ if and only if

$$\sum_{j=1}^{N}\operatorname*{Res}_{\xi_j=0}(g_j(\xi_j)\tau_j)=0$$

for any $\tau\in H^0(C,\omega_C(m\sum_{j=1}^N Q_j))$. The proof of (1.51) will be given later.

Assume that $(f_j(\xi_j)) \in \bigoplus_{j=1}^N \mathbb{C}((\xi_j))$ is annihilated by $H^0(C, \omega_C(*\sum_{j=1}^N Q_j))$ by the residue pairing. Fix a positive integer m. For

$$f_j(\xi_j) = \sum_{k=-n_j}^{\infty} a_k^{(j)} \xi_j^k, \quad j = 1, 2, \ldots, N$$

put

$$f_{j,m}(\xi_j) = \sum_{k=-n_j}^{m-1} a_k^{(j)} \xi_j^k, j = 1, 2, \ldots, N.$$

Put $n = \max_j n_j$. Since $(f_j(\xi_j))$ is annihilated by $H^0(C, \omega_C(*\sum_{j=1}^N Q_j))$, $(f_{j,m}(\xi_j))$ is annihilated by $H^0(C, \omega_C(m\sum_{j=1}^N Q_j))$. $(f_{j,m}(\xi_j))$ is the image of a meromorphic function $f^{(m)} \in H^0(C, \mathcal{O}_{\mathcal{C}}(n\sum_{j=1}^N Q_j))$. Let $f_j^{(m)}(\xi_j)$ be the Laurent expansion of $f^{(m)}$ at Q_j with respect to ξ_j. Then

$$g_j(\xi_j) := f_j(\xi_j) - f^{(m)}(\xi_j) \equiv 0 \bmod \xi_j^m.$$

Since $f^{(m)}$ is a global meromorphic function $(f_j^{(m)}(\xi_j))$ is annihilated by

$$H^0(C, \omega_C(*\sum_{j=1}^N Q_j)).$$

$(g_j(\xi_j))$ is also annihilated by $H^0(C, \omega_C(*\sum_{j=1}^N Q_j))$. Assume that $g_k(\xi_k) \neq 0$ so that

$$g_k(\xi_j) = b_s \xi_k^s + b_{s+1}\xi_k^{s+1} + \cdots, \quad b_s \neq 0.$$

Then $s \geq m > 0$. Hence there exists a meromorphic one-form

$$\omega \in H^0(C, \omega_C((s+1)Q_k)$$

which has a pole of order $s+1$ at Q_k and is holomorphic on $C \setminus Q_k$. Hence we have

$$\sum_{j=1}^N \operatorname*{Res}_{\xi_j=0} g_j\omega_j = \operatorname*{Res}_{\xi_k=0} g_k\omega_k = b_s \neq 0,$$

where ω_j is the Laurent expansion of ω at Q_j with respect to ξ_j. This contradicts the fact that $\{g_j(\xi_j)\}$ is annihilated by $H^0(C, \omega_C(m\sum_{j=1}^N Q_j))$. Thus we conclude that $g_j(\xi_j) = 0$, $j = 1, 2, \ldots, N$. Thus $f_j(\xi_j)$ is the Laurent expansion of $f^{(m)}$ at Q_j with respect to ξ_j.

Let us show (1.51). First assume that C is a compact Riemann surface. For a small positive number ε put $D_\varepsilon^{(j)} = \{\, \xi_j \,|\, |\xi_j| < \varepsilon \,\}$ and $C_\varepsilon = C \setminus \sum_{j=1}^N D_\varepsilon^{(j)}$. Cut C_ε along a closed curve so that the cut surface R'_ε is simply connected (see Figure 1.5).

Choose an open covering $\{U_i\}_{i=1}^t$ of C such that

$$Q_j \in U_j, \quad j = 1, 2, \ldots, N, \quad Q_j \notin U_i, \quad i \neq j.$$

Choose U_i small enough so that for $j = 1, \ldots, N$ there exist

$$h_j \in H^0(U_j, \mathcal{O}_{\mathcal{C}}(m\sum_{j=1}^N Q_j))$$

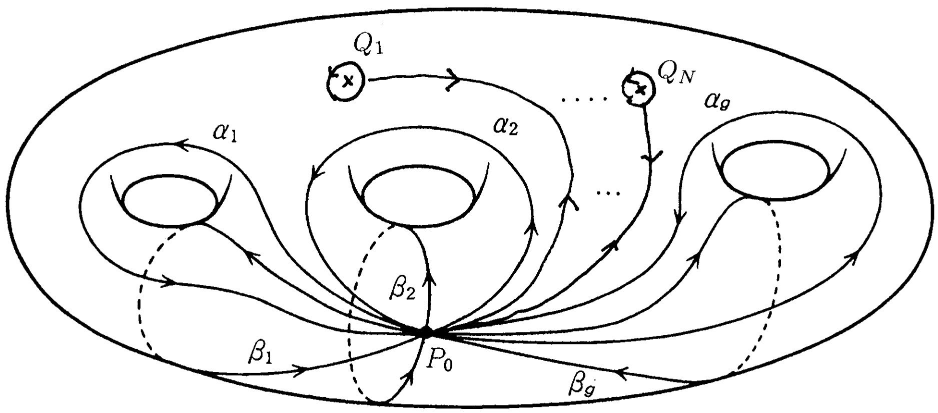

Figure 1.5 Cut the surface C_ε along a closed curve.

with $p(h_j) = g_j(\xi_j)$. Set $h_i = 0$ for $i \geq N+1$. Then,

$$h_{ab} = h_b - h_a \in H^0(U_a \cap U_b, \mathcal{O}_{\mathcal{C}}(-n \sum_{j=1}^{N} Q_j))$$

and $\{h_{ab}\}$ is a one-cocycle which represents

$$c((g_j(\xi_j)) \in H^1(C, \mathcal{O}_{\mathcal{C}}(-n \sum_{j=1}^{N} Q_j)).$$

The duality pairing $H^1(C, \mathcal{O}_{\mathcal{C}}(-m \sum_{j=1}^{N} Q_j)) \times H^0(C, \mathcal{O}_{\mathcal{C}}(n \sum_{j=1}^{N} Q_j)) \to \mathbb{C}$ is given by the integration

$$\frac{1}{2\pi i} \int_C \omega \wedge \tau$$

where $\omega \in H_{\overline{\partial}}^{(0,1)}(C, \mathcal{O}_{\mathcal{C}}(-m \sum_{j=1}^{N} Q_j))$ is a Dolbeault cohomology class. Let $\mathcal{D}_C^{(0,0)}$ (resp. $\mathcal{D}_C^{(0,k)}$) be the sheaf of germs of complex-valued C^∞ functions (resp. type $(0,k)$-forms) on C and define

$$\mathcal{D}_C^{(0,i)}(-n \sum_{j=1}^{N} Q_j) = \mathcal{O}_{\mathcal{C}}(-n \sum_{j=1}^{N} Q_j) \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{D}_C^{(0,i)}.$$

Then we have

$$\begin{aligned} & H_{\overline{\partial}}^{(0,1)}(C, \mathcal{O}_{\mathcal{C}}(-m \sum_{j=1}^{N} Q_j)) \\ \simeq \quad & \frac{\operatorname{Ker}\left\{\overline{\partial} : H^0(C, \mathcal{D}_C^{(0,1)}(-m \sum_{j=1}^{N} Q_j)) \to H^0(C, \mathcal{D}_C^{(0,2)}(-m \sum_{j=1}^{N} Q_j))\right\}}{\overline{\partial}\left(H^0(C, \mathcal{D}_C^{(0,0)}(-m \sum_{j=1}^{N} Q_j))\right)}. \end{aligned}$$

The Dolbeault cohomology class associated with $\{h_{ab}\} = c(g_j(\xi_j))$ is given as follows. There exist $s_a \in H^0(U_a, \mathcal{D}_C^{(0,0)}(-n\sum_{j=1}^N Q_j))$ such that

$$h_{ab} = s_b - s_a.$$

Then we have

$$h_a - s_a = h_b - s_a.$$

for any a, b so that they define a global function h on C. Note that h has a pole at Q_j so that $\overline{\partial} h$ is not the zero cohomology class. The class $\overline{\partial} h$ represents the Dolbeault cohomology class associated with the Čech one-cocycle $\{h_{ab}\}$.

$$\overline{\partial} h = -\overline{\partial} s_a$$

on U_a. Then the pairing $\langle c((g_j(\xi_j)), \tau\rangle$ is calculated as follows.

$$\begin{aligned}
\langle c((g_j(\xi_j)), \tau\rangle &= \frac{1}{2\pi i}\int_C \overline{\partial} h \wedge \tau \\
&= \frac{1}{2\pi i}\int_C d(h\tau) \\
&= \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{C_\varepsilon} d(h\tau) \\
&= \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{\partial C} h\tau \\
&= \lim_{\varepsilon\to 0} \sum_{j=1}^N \frac{1}{2\pi i}\int_{\partial D_\epsilon^{(j)}} (h_j - s_j)\tau \\
&\qquad \text{(integrations along } \alpha_i,\ \beta_i \text{ and } l_j \text{ cancel out)} \\
&= \lim_{\varepsilon\to 0} \sum_{j=1}^N \frac{1}{2\pi i}\left\{\int_{\partial D_\epsilon^{(j)}} h_j\tau - \int_{\partial D_\epsilon^{(j)}} s_j\tau\right\} \\
&= \sum_{j=1}^N \operatorname*{Res}_{Q_j}(h_j\tau) - \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{D_\epsilon^{(j)}} d(s_j\tau).
\end{aligned}$$

Since $s_j\tau$ is C^∞ in a neighbourhood of Q_j, we have

$$\lim_{\varepsilon\to 0}\int_{D_\epsilon^{(j)}} d(s_j\tau) = 0.$$

Also, we have $\operatorname{Res}_{Q_j}(h_j\tau) = \operatorname{Res}_{\xi_j=0}(g_j(\xi_j)\tau_j)$ where τ_j is the Laurent expansion of τ at Q_j with respect to ξ_j; hence we conclude

$$\langle c((g_j(\xi_j)), \tau\rangle = \sum_{j=1}^N \operatorname*{Res}_{\xi_j=0} g_j(\xi_j)\tau_j.$$

This is the desired result.

Next assume that C has nodes. For simplicity assume that C has only one node P. Let $\pi : \widetilde{C} \to C$ be the normalization map and $\pi^{-1}(P) = \{P_+, P_-\}$. Then $\pi^*\omega_C = \omega_{\widetilde{C}}(P_+ + P_-)$. Hence

$$\pi^* H^0(C, \omega_C(m\sum_{j=1}^N Q_j)) \subset H^0(\widetilde{C}, \omega_{\widetilde{C}}(m\sum_{j=1}^N Q_j)).$$

This implies that on $\widetilde{C}$, $(g_j(\xi_j))$ is annihilated by $H^0(\widetilde{C}, \omega_{\widetilde{C}}(m\sum_{j=1}^N Q_j))$. Hence, by the above arguments there exists $\tilde{g} \in H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(m\sum_{j=1}^N Q_j))$ such that $p(\tilde{g}) = (g_j(\xi_j))$. On the other hand $(g_j(\xi_j))$ is annihilated by $H^0(C, \omega_C(m\sum_{j=1}^N Q_j))$. Hence for $\tau \in \pi^* H^0(C, \omega_C(m\sum_{j=1}^N Q_j)) \setminus H^0(\widetilde{C}, \omega_{\widetilde{C}}(m\sum_{j=1}^N Q_j))$ we have

$$\sum_{j=1}^N \operatorname*{Res}_{Q_j}(\tilde{g}\tau) = 0.$$

On the other hand $\tilde{g}\tau$ has poles of order one at P_+ and P_-. Therefore we conclude

$$\operatorname*{Res}_{P_+}(\tilde{g}\tau) + \operatorname*{Res}_{P_-}(\tilde{g}\tau) = 0. \tag{1.52}$$

Since we have

$$\operatorname*{Res}_{P_+}(\tilde{g}\tau) = \tilde{g}(P_+)\operatorname*{Res}_{P_+}\tau, \quad \operatorname*{Res}_{P_-}(\tilde{g}\tau) = \tilde{g}(P_-)\operatorname*{Res}_{P_-}\tau$$

and

$$\operatorname*{Res}_{P_+}\tau + \operatorname*{Res}_{P_-}\tau = 0,$$

we have

$$\tilde{g}(P_+) = \tilde{g}(P_-).$$

This means that $\tilde{g}$ is the pullback of a meromorphic function

$$g \in H^0(C, \mathcal{O}_{\mathcal{C}}(m\sum_{j=1}^N Q_j)).$$

This shows that (1.51) is valid for a nodal curve.

The characterization of $H^0(C, \omega_C(*\sum_{j=1}^N Q_j))$ is proved similarly. □

1.3 Deformation of pointed curves

Let C be a compact Riemann surface of genus g. A deformation of C is, by definition, a proper *smooth* holomorphic mapping $\pi : X \to Y$ of complex spaces with a prescribed point $y \in Y$ such that $\pi^{-1}(y)$ is isomorphic to the Riemann surface C (see 1.1.2). It is more convenient to generalize the notion of complex analytic family $f : X \to Y$ in such a way that the underlying space Y is a complex space, not necessarily a complex manifold.

For example, if $Y = \operatorname{Specan}\mathbb{C}[\,\epsilon\,]/(\epsilon^2)$, the deformation of C is called an infinitesimal (or first oder) deformation of C. The infinitesimal deformations of the Riemann surface C are parameterized by the cohomology group $H^1(C, \Theta_C)$, where Θ_C is the sheaf of germs of holomorphic vector fields on C. (See 1.1.2.)

More generally, we can define a deformation of the data

$$\mathfrak{X}^{(n)} = (C\,;\, Q_1, Q_2, \ldots, Q_N\,;\, \eta_1^{(n)}, \eta_2^{(n)}, \ldots, \eta_N^{(n)})$$

of an N-pointed Riemann surface of genus g with n-th infinitesimal neighbourhoods. The infinitesimal deformations of an N-pointed Riemann surface of genus g with n-th infinitesimal neighbourhoods are parameterized by the cohomology group $H^1(C, \Theta_C(-(n+1)\sum_{j=1}^N Q_j)$. If C is a singular semi-stable curve, a deformation of C is defined as a proper *flat* holomorphic mapping $\pi : X \to Y$ of complex spaces with a prescribed point $y \in Y$ such that $\pi^{-1}(y)$ is isomorphic to the curve C. In this case, the infinitesimal deformations of the curve C are parameterized by the cohomology group $Ext^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$ and the infinitesimal deformations of a stable N-pointed curve

$$\mathfrak{X}^{(n)} = (C\,;\,Q_1, Q_2, \dots, Q_N\,;\,\eta_1^{(n)}, \eta_2^{(n)}, \dots, \eta_N^{(n)})$$

with n-th infinitesimal neighbourhoods are parameterized by the cohomology group

$$Ext^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C(-(n+1)\sum_{j=1}^N Q_j).$$

(See, for example, [Ar], [DM, Section 1], [SGA7, Exposé VI, 6].) Here, Ω^1_C is the sheaf of Kähler differentials on the curve C. In our situation, we may regard the exact sequence

$$0 \to \mathcal{I}_C/\mathcal{I}_C^2 \to \Omega^1_Y \otimes \mathcal{O}_C \to \Omega^1_C \to 0.$$

as a definition of the sheaf Ω^1_C where $\mathcal{I}_C$ is the sheaf of defining ideals of $C = f^{-1}(y_0)$ of a deformation $\pi : X \to Y$ of the curve C.) Put $\Theta_C = \underline{Hom}_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$. There is an exact sequence

$$\begin{aligned} 0 \to H^1(C, \Theta_C(-(n+1)\sum_{j=1}^N Q_j)) \quad &\to \quad Ext^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C(-(n+1)\sum_{j=1}^N Q_j)) \\ &\to \quad H^0(C, \underline{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)) \to 0. \end{aligned} \tag{1.53}$$

If the stable curve C has q double points $P_1, P_2, \dots, P_q$, then we have

$$\underline{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)_Q = \begin{cases} \mathbb{C}, & \text{if } Q = P_j, \quad i = 1, 2, \dots, q \\ 0, & \text{otherwise.} \end{cases}$$

Hence we have

$$H^0(C, \underline{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)) \simeq \mathbb{C}^q.$$

Each element of $H^1(C, \Theta_C(-(n+1)\sum_{j=1}^N Q_j))$ corresponds to an infinitesimal deformation of the data

$$\mathfrak{X}^{(n)} = (C\,;\,Q_1, Q_2, \dots, Q_N\,;\,\eta_1^{(n)}, \eta_2^{(n)}, \dots, \eta_N^{(n)})$$

preserving the singularities.

Definition 1.23 The data $(\pi : Y \to B; s_1, s_2, \dots, s_N; \eta_1, \eta_2, \dots, \eta_N)$ is called a (holomorphic) family of N-pointed stable curves of genus g with formal neighbourhoods if the following conditions are satisfied.

(1) Y and B are connected complex manifolds, $\pi : Y \to B$ is a proper flat holomorphic map and $s_1, s_2, \dots, s_N$ are holomorphic sections of π.
(2) For each point $b \in B$ the data $(Y_b := \pi^{-1}(b); s_1(b), s_2(b), \dots, s_N(b))$ is an N-pointed stable curve of genus g.
(3) η_j is an $\mathcal{O}_B$-algebra isomorphism

$$\eta_j : \widehat{\mathcal{O}}/s_j(B) := \varprojlim_{n\to\infty} \mathcal{O}_Y/I_j^n \simeq \mathcal{O}_B[[\xi]],$$

where I_j is the defining ideal of $s_j(B)$ in Y.

Similarly we define a family of stable N-pointed curves of genus g

$$(\pi : Y \to B; s_1, s_2, \dots, s_N; \tilde{\eta}_1^{(n)}, \tilde{\eta}_2^{(n)}, \dots, \tilde{\eta}_N^{(n)})$$

with n-th formal neighbourhoods, by changing the third condition by

(3′) $\tilde{\eta}_j^{(n)}$ is an $\mathcal{O}_B$-algebra isomorphism

$$\tilde{\eta}_j^{(n)} : \mathcal{O}_Y/I_j^{n+1} \simeq \mathcal{O}_B[[\xi]]/I_j^{(n+1)}.$$

For a holomorphic mapping $f : D \to B$, we can define the pullback $f^*\mathfrak{X}^{(n)}$ (resp. $f^*\mathfrak{X}$) of the family of N-pointed stable curves with n-th infinitesimal neighbourhoods (resp. formal neighbourhoods) over the base space D by the mapping f.

Definition 1.24 A family

$$\mathfrak{X} = (\,\pi^{(n)} : \mathcal{C}^{(n)} \to \mathcal{B}^{(n)}\,;\; s_1^{(n)}, s_2^{(n)}, \ldots, s_N^{(n)}\,;\; \tilde{\eta}_1^{(n)}, \tilde{\eta}_2^{(n)}, \ldots, \tilde{\eta}_N^{(n)})$$

of stable N-pointed curves of genus g with n-th formal neighbourhoods is said to be *versal* (resp. *universal*) at a point $b \in \mathcal{B}^{(n)}$ if for any deformation

$$\mathfrak{Y} = (\pi : X \to Y; s_1, \ldots, s_N; \widehat{\eta}_1, \ldots, \widehat{\eta}_N)$$

of $\pi^{(n)-1}(b) = (C; Q_1, \ldots, Q_N; \eta_1, \ldots, \eta_N)$ with prescribed point $y \in Y$ there exists a holomorphic mapping (resp. unique holomorphic mapping) f from a neighbourhood of y in Y to $\mathcal{B}^{(n)}$ such that the pullback $f^*\mathfrak{X}$ is isomorphic to $\mathfrak{Y}$ in a neighbourhood of y in Y and such that df is uniquely determined at the base point. If the family is versal (resp. universal) at each point of $\mathcal{B}^{(n)}$, the family $\mathfrak{X}$ is called a versal (resp. universal) family.

In the following we mainly consider a family of stable pointed curves with formal coordinates, which is versal as a family of pointed curves. Hence, in this section we only consider a family of stable pointed curves. The Kodaira-Spencer mapping of a family of stable pointed curves is given by the following theorem. For a proof see [TUY, p. 499–503].

Theorem 1.25 *For a family $(\pi : Y \to B; s_1, s_2, \ldots, s_N)$ of N-pointed stable curve of genus g and for each point $b \in B$, there exists a $\mathbb{C}$-linear mapping*

$$\rho_b : T_bB \to Ext^1_{\mathcal{O}_{Y_b}}(\Omega^1_{Y_b}, \mathcal{O}_{Y_b}(-\sum_{j=1}^{N} s_j(b)))\,, \tag{1.54}$$

where $Y_b = \pi^{-1}(b)$.

The $\mathbb{C}$-linear mapping ρ_b is called the *Kodaira-Spencer mapping* of the family $(\pi : Y \to B; s_1, s_2, \ldots, s_N)$ at the point b.

A sheaf version of Theorem 1.25 is the following:

Corollary 1.26 *If $(\pi : Y \to B; s_1, s_2, \ldots, s_N)$ is a family of N-pointed* smooth *curves of genus g, the Kodaira-Spencer mapping ρ_s induces an $\mathcal{O}_B$-module homomorphism*

$$\rho : \Theta_B \to R^1\pi_*\underline{Hom}(\Omega^1_{Y/B}, \Theta_Y(-\sum_{j=1}^{N} s_j(B))).$$

The criterion of versality of a family is given by the following:

Proposition 1.27 *A family*

$$(\,\pi : \mathcal{C} \to \mathcal{B}\,;\; s_1, s_2, \ldots, s_N)$$

of stable N-pointed curves of genus g is versal at a point $b \in \mathcal{B}$ if and only if the Kodaira-Spencer mapping

$$\rho_b : T_b B \to Ext^1_{\mathcal{O}_{C_b}}(\Omega^1_{\mathcal{O}_{C_b}}, \mathcal{O}_{C_b}(-\sum_{j=1}^{N} s_j(b)))$$

is an isomorphism at the point b.

Let us prove the existence of the versal family of stable N-pointed curves.

Theorem 1.28 *For each N-pointed stable curve $(C; Q_1, \ldots, Q_N)$ of genus g, there exists a family*

$$\mathfrak{F} = (\,\pi : \mathcal{C} \to \mathcal{B}\,;\, s_1, s_2, \ldots, s_N)$$

with prescribed point $b \in \mathcal{B}$ such that $\pi^{-1}(b)$ is isomorphic to $(C; Q_1, \ldots, Q_N)$ and such that the family $\mathfrak{F}$ is versal at b. Moreover, $\mathcal{C}$ and $\mathcal{B}$ are complex manifolds and the family $\mathfrak{F}$ is versal at each point of a small neighbourhood of b in $\mathcal{B}$. If the N-pointed stable curve has trivial automorphism group, then the family $\mathfrak{F}$ is universal at b.

Proof The theorem is a consequence of deformation theory ([Ar], [Sc], [SGA 7]). Since we need an explicit description of a versal family, we give here a method to construct a versal family of the first infinitesimal neighbourhoods.

By our assumption, the curve C has only ordinary double points. Hence, by deformation theory, there exists a versal family $\pi^{(\phi)} : \mathcal{C}^{(\phi)} \to \mathcal{B}^{(\phi)}$ with specified point $x \in \mathcal{B}^{(\phi)}$ such that $C_x = \pi^{(\phi)-1}(x) \simeq C$ and the Kodaira-Spencer mapping

$$\rho_x : T_x\mathcal{B}^{(\phi)} \to Ext^1_{\mathcal{O}_{C_x}}(\Omega^1_{C_x}, \mathcal{O}_{C_x})$$

is an isomorphism. (Since the automorphism group of C may not be trivial, the family $\pi : \mathcal{C}^{(\phi)} \to \mathcal{B}^{(\phi)}$ may not be universal at the point x but only versal.) Put

$$\mathcal{B} = \mathcal{C}^{(\phi)N} \setminus \left(\bigcup_{i<j} \Delta_{ij} \bigcup \{ \text{ singular points of } \mathcal{C}^{(\phi)}N \ \} \right)$$

where

$$\Delta_{ij} = \{ \ (x_1, \ldots, x_N) \in \mathcal{C}^{(\phi)N} \mid \ x_i = x_j \ \}$$

is the (i,j)-th diagonal. There is a natural holomorphic mapping $p : \mathcal{B} \to \mathcal{B}^{(\phi)}$. Put also

$$\mathcal{C} = \mathcal{C}^{(\phi)} \times_{\mathcal{B}^{(\phi)}} \mathcal{B}$$

and let $\pi : \mathcal{C} \to \mathcal{B}$ be the projection to the second factor. By our definition, $(Q_1, \ldots, Q_N) \in p^{-1}(x)$. Put $x_0 = (Q_1, \ldots, Q_N) \in \mathcal{B}$. Then we have $\pi^{-1}(x_0) = C_x \times x_0 \simeq C$. Moreover, we can define holomorphic sections

$$s_j : \mathcal{B} \to \mathcal{C}$$

by

$$s_j((P_1, \ldots P_N)) = (P_j, P_1, \ldots, P_N) \in \mathcal{C}^{(\phi)} \times_{\mathcal{B}^{(\phi)}} \mathcal{B}.$$

Then we have $s_j(x_0) = (Q_j, x_0)$. It is easy to show that $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_N)$ is versal at each point of $\mathcal{B}$. □

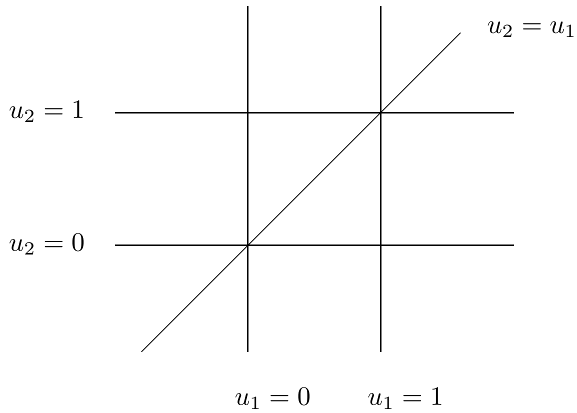

Figure 1.6 The moduli space of ordered four-pointed projective lines : $B_2 = \mathbb{C}^2 \setminus \{4 \text{ lines}\}$.

Let $\mathfrak{F}(\,\pi := \mathcal{C} \to \mathcal{B}\,;\, s_1, s_2, \dots, s_N)$ be a versal family of N-pointed stable curves of genus g. Put

$$\Sigma \;=\; \{\, P \in \mathcal{C} \,|\, d\pi_P : T_P\mathcal{C} \to T_{\pi(P)}\mathcal{B} \text{ is not surjective} \,\} \tag{1.55}$$

$$D \;=\; \pi(\Sigma). \tag{1.56}$$

The set Σ is called the *critical locus* of the family and D is called the *discriminant locus* of the family. The following lemma is a consequence of the deformation theory of singular curves with ordinary double points. (See for example [Ar], [DM, Section 1] or [SGA 7, Exposé VI, 6].)

Lemma 1.29 *For a versal family of stable N-pointed curves of genus g*

$$(\,\pi : \mathcal{C} \to \mathcal{B}\,;\, s_1, s_2, \dots, s_N),$$

assume $2g - 2 + N \geq 1$.

(1) *We have*

$$\dim \mathcal{B} \;=\; 3g - 3 + N$$
$$\dim \mathcal{C} \;=\; 3g - 2 + N.$$

(2) *The critical locus Σ is a smooth subvariety of codimension 2 in $\mathcal{C}$.*
(3) *The discriminant locus D is a divisor with normal crossings in $\mathcal{B}$.*

Example 1.30 Let $(\mathbb{P}^1; Q_1, \dots, Q_N)$ be an N-pointed projective line (smooth curve of genus 0). By an automorphism of $\mathbb{P}^1$, the first three points Q_1, Q_2, Q_3 can be mapped to the points 0, 1, ∞. We may regard $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$. Let u be a global coordinate of $\mathbb{C}$. Thus, $(\mathbb{P}^1; Q_1, \dots, Q_N)$ is isomorphic to $(\mathbb{P}^1; 0, 1, \infty, u_1, \dots, u_{N-3})$. Put

$$B_m = \{(u_1, \dots, u_m) \in \mathbb{C} \mid u_j \neq 0, 1, \infty, \;\; u_i \neq u_j, \;\; i \neq j\}.$$

For example, $B_1 = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and $B_2 = \mathbb{C}^2 \setminus$ the lines in the figure 1.6.

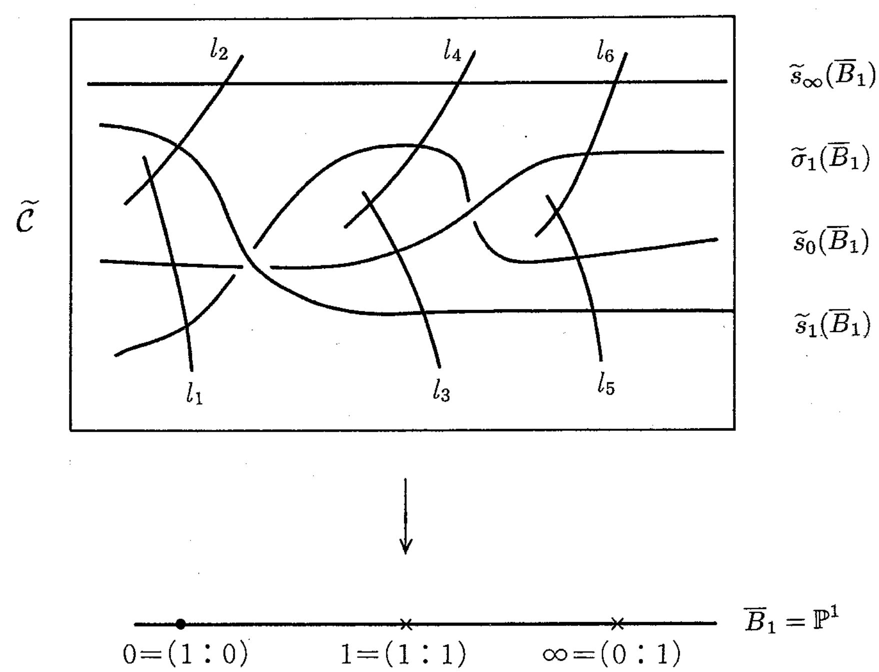

Figure 1.7 The universal family of stable four-pointed curves of genus 0.

Let $\pi : \mathcal{C} = \mathbb{P}^1 \times B_m \to B_m$ be the projection to the second factor. For any point $(u_1, \ldots, u_m) \in B_m$ put

$$\begin{aligned} s_k((u_1, \ldots, u_m)) &= k, \quad k = 0, 1, \infty \\ \sigma_i((u_1, \ldots, u_m)) &= u_i, \quad i = 1, 2, \ldots, m. \end{aligned}$$

Note that s_k and σ_i are holomorphic sections of π. Then B_m is the moduli space of ordered $(m+3)$-pointed projective lines and $(\pi : \mathcal{C} \to B_m; s_0, s_1, s_\infty; \sigma_1, \ldots, \sigma_m)$ is the universal family of ordered $(m+3)$-pointed projective lines. The extension of the universal family to that of stable $(m+3)$-pointed curves of genus 0 was first studied by Terada [T] in a different context. The detailed study of the moduli space of stable N-pointed curves of genus 0 was done by Gerritzen, Herrlich and van der Put [GHP]. In the case $m = 1$ the extension can be easily described. The compactification $\overline{B}_1$ of B_1 is $\mathbb{P}^1$. Let $\widetilde{\mathcal{C}}$ be the blowing up of $\mathbb{P}^1 \times \overline{B}_1$ at the points $(0, 0)$, $(1, 1)$ and (∞, ∞) and $\widetilde{\pi} : \widetilde{\mathcal{C}} \to \overline{B}_1$ be the natural holomorphic mapping. Then there are natural extensions $\widetilde{s}_k$ and $\widetilde{\sigma}_i$ of s_k and σ_1, respectively, which are holomorphic sections of $\widetilde{\pi}$. Then, $(\widetilde{\pi} : \widetilde{\mathcal{C}} \to \overline{B}_1; \widetilde{s}_0, \widetilde{s}_1, \widetilde{s}_\infty, \widetilde{\sigma}_1)$ is the universal family of ordered stable four-pointed curves of genus 0 (1.30). The fibres of $\widetilde{\pi}$ over the compactified points 0, 1, ∞ are stable four-pointed curves of genus 0 with ordinary double point. This example will be considered in §5.4 below.

By Knudsen [Kn], the compactification $\overline{B}_2$ of B_2 by means of ordered five-pointed stable curves of genus 0 is nothing but our $\widetilde{\mathcal{C}}$. For the explicit description by means of coordinates we refer the reader to [T] and [GHP].

1.4 Versal family of stable pointed curves

Let us consider a versal family $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}\,;\, s_1, s_2, \dots, s_N)$ of stable N-pointed curves of genus g. We assume that we have a local (formal) coordinate η with center $s_N(\mathcal{B})$. In the following we need to consider locally a family $\mathfrak{F}$. For that purpose we introduce the following local coordinates of $\mathcal{C}$.

For a point $P \in \Sigma$ of the critical locus of π, we can choose local coordinates $(u_1, u_2, \dots, u_{M-1}, z, w)$ of $\mathcal{C}$ with center P and local coordinates $(\tau_1, \tau_2, \dots, \tau_M)$ of $\mathcal{C}$ with center $\pi(P)$ such that the holomorphic mapping π is given by

$$(u_1, u_2, \dots, u_{M-1}, z, w) \longmapsto (u_1, u_2, \dots, u_{M-1}, zw) = (\tau_1, \tau_2, \dots, \tau_M)\,.$$

In other words, we have

$$\pi^*\tau_k = \begin{cases} u_k\,, & k = 1, 2, \dots, M-1 \\ zw\,, & k = M\,. \end{cases}$$

For a point $P \in \mathcal{C} \setminus \Sigma$ we can choose local coordinates

$$(u_1, u_2, \dots, u_M, z)$$

of $\mathcal{C}$ with center P and local coordinates $(\tau_1, \tau_2, \dots, \tau_M)$ of $\mathcal{C}$ with center $\pi(P)$ such that the holomorphic mapping is given by

$$(u_1, u_2, \dots, u_{M-1}, z) \longmapsto (u_1, u_2, \dots, u_M) = (\tau_1, \tau_2, \dots, \tau_M)\,.$$

An $\mathcal{O}_\mathcal{C}$-module $\Omega^1_{\mathcal{C}/\mathcal{B}}$ is defined by the exact sequence

$$\pi^{-1}\Omega^1_\mathcal{B} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_\mathcal{C} \to \Omega^1_\mathcal{C} \to \Omega^1_{\mathcal{C}/\mathcal{B}} \to 0.$$

The sheaf $\Omega^1_{\mathcal{C}/\mathcal{B}}$ is called the sheaf of germs of relative 1-forms of the family $\pi : \mathcal{C} \to \mathcal{B}$. Let us describe the sheaf $\Omega^1_{\mathcal{C}/\mathcal{B}}$ by using the above local coordinates. In a neighbourhood of a point $P \in \mathcal{C} \setminus \Sigma$, the sheaf $\Omega^1_{\mathcal{C}/\mathcal{B}}$ is locally isomorphic to $\mathcal{O}_\mathcal{C} dz$. In a small neighbourhood of $P \in \Sigma$, we have an $\mathcal{O}_\mathcal{C}$-module isomorphism

$$\Omega^1_{\mathcal{C}/\mathcal{B}} \simeq (\mathcal{O}_\mathcal{C} dz + \mathcal{O}_\mathcal{C} dw)/\mathcal{O}_\mathcal{C}(wdz + zdw)\,. \tag{1.57}$$

Moreover, we have the following lemma.

Lemma 1.31 *The following sequence*

$$0 \to \pi^{-1}\Omega^1_\mathcal{B} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_\mathcal{C} \to \Omega^1_\mathcal{C} \to \Omega^1_{\mathcal{C}/\mathcal{B}} \to 0 \tag{1.58}$$

is exact and gives a locally free resolution of the sheaf $\Omega^1_{\mathcal{C}/\mathcal{B}}$.

Let $\omega_{\mathcal{C}/\mathcal{B}}$ be the relative dualizing sheaf of $\pi : \mathcal{C} \to \mathcal{B}$. Since $\mathcal{C}$ and $\mathcal{B}$ are non-singular and π is flat, we have an $\mathcal{O}_\mathcal{C}$-module isomorphism

$$\omega_{\mathcal{C}/\mathcal{B}} \simeq \omega_\mathcal{C} \otimes (\pi^*\omega_\mathcal{B}^{-1})$$

where ω_Y is the dualizing sheaf (canonical sheaf) of a complex manifold Y. The relative dualizing sheaf $\omega_{\mathcal{C}/\mathcal{B}}$ is described locally as follows. (See also (1.44).)

In a small neighbourhood of a point $P \in \mathcal{C} \setminus \Sigma$, we have

$$\omega_{\mathcal{C}/\mathcal{B}} = \Omega^1_{\mathcal{C}/\mathcal{B}} \simeq \mathcal{O}_\mathcal{C} dz\,.$$

In a small neighbourhood of a point $P \in \Sigma$, we have

$$\omega_{\mathcal{C}/\mathcal{B}} \simeq \mathcal{O}_\mathcal{C}(dz \wedge dw) \otimes (d\tau_M)^{-1}\,.$$

In particular, we have

$$\omega_{\mathcal{C}/\mathcal{B}} \simeq \begin{cases} \mathcal{O}_{\mathcal{C}} \dfrac{dz}{z} & \text{if } z \neq 0 \\ \\ \mathcal{O}_{\mathcal{C}} \dfrac{dw}{w} & \text{if } w \neq 0 \end{cases}$$

with relation

$$\frac{dz}{z} + \frac{dw}{w} = 0$$

if $zw \neq 0$.

Lemma 1.32 *There exists an exact sequence*

$$0 \to \Omega^1_{\mathcal{C}/\mathcal{B}} \to \omega_{\mathcal{C}/\mathcal{B}} \to \omega_{\mathcal{C}/\mathcal{B}} \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{O}_{\Sigma} \to 0\,.$$

Proof The mapping $\Omega^1_{\mathcal{C}/\mathcal{B}} \to \omega_{\mathcal{C}/\mathcal{B}}$ is given locally in a neighbourhood of a point $P \in \mathcal{C} \setminus \Sigma$ by

$$dz \longmapsto dz$$

and in a neighbourhood of a point $P \in \Sigma$ by

$$\begin{array}{rcl} dz & \longmapsto & z(dz \wedge dw) \otimes (d\tau_M)^{-1} \\ dw & \longmapsto & w(dz \wedge dw) \otimes (d\tau_M)^{-1}\,. \end{array}$$

In particular, we have

$$\begin{array}{rcll} dz & \longmapsto & z\dfrac{dz}{z} & \text{if } z \neq 0 \\ dw & \longmapsto & w\dfrac{dw}{w} & \text{if } w \neq 0\,. \end{array}$$

This proves Lemma 1.32. □

Lemma 1.33 *Put*

$$\Theta_{\mathcal{C}/\mathcal{B}} = \underline{Hom}_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}). \tag{1.59}$$

Then $\Theta_{\mathcal{C}/\mathcal{B}}$ is an invertible $\mathcal{O}_{\mathcal{C}}$-module and there is an isomorphism

$$\Theta_{\mathcal{C}/\mathcal{B}} \simeq \underline{Hom}_{\mathcal{O}_{\mathcal{C}}}(\omega_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}). \tag{1.60}$$

Hence, $\Theta_{\mathcal{C}/\mathcal{B}}$ is an invertible sheaf.

Proof By (1.58) it is easy to show that in a neighbourhood of a point $P \in \mathcal{C} \setminus \Sigma$ we have an isomorphism

$$\Theta_{\mathcal{C}/\mathcal{B}} \simeq \mathcal{O}_{\mathcal{C}} \frac{\partial}{\partial z}$$

and in a neighbourhood of a point $P \in \Sigma$ we have an isomorphism

$$\Theta_{\mathcal{C}/\mathcal{B}} \simeq \mathcal{O}_{\mathcal{C}}(z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}). \tag{1.61}$$

By this fact and (1.57), we have the desired result. □

From the exact sequence (1.58) we obtain the following Corollary.

Corollary 1.34 *The following sequence*

$$0 \to \Theta_{\mathcal{C}/\mathcal{B}} \to \Theta_{\mathcal{C}} \to \pi^{-1}\Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}} \to \underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}) \to 0$$

of $\mathcal{O}_{\mathcal{C}}$-modules is exact.

Lemma 1.35 *There exists an exact sequence*

$$0 \to \Theta_{\mathcal{B}}(-\log D) \to \Theta_{\mathcal{B}} \xrightarrow{t} \pi_* \underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}) \to 0 \tag{1.62}$$

where

$$\Theta_{\mathcal{B}}(-\log D) = \{ \, v \in \Theta_{\mathcal{B}} \mid \; v(\mathcal{I}_D) \subset \mathcal{I}_D \, \}$$

and $\mathcal{I}_D$ is the sheaf of defining ideals of D in $\mathcal{B}$.

Proof First note that the sheaf $\Theta_{\mathcal{B}}(-\log D)$ is a sheaf of germs of vector fields on $\mathcal{B}$ tangent to D. Since $\pi : \mathcal{C} \to \mathcal{B}$ is a versal family, using the Kodaira-Spencer mapping and (1.53), for each point $s \in \mathcal{B}$ we have an exact sequence

$$0 \to H^1(C_s, \Theta_{C_s}(-\sum_{j=1}^N s_j(s))) \to T_s\mathcal{B} \to H^0(C_s, \underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega^1_{C_s}, \mathcal{O}_{C_s})) \to 0.$$

Each element of $H^1(C_s, \Theta_{C_s}(-\sum_{j=1}^N s_j(s)))$ corresponds to a tangent vector of $\mathcal{B}$ at s preserving the singularities of C_s. Hence the sheaf version of the above sequence is also exact. □

Theorem 1.36 *Let $(\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \ldots, s_N)$ be a versal family of stable N-pointed curves of genus g. Then there exists an $\mathcal{O}_{\mathcal{B}}$-module isomorphism*

$$\rho : \Theta_s(-\log D) \xrightarrow{\sim} R^1\pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) \tag{1.63}$$

where we put $S_j = s_j(\mathcal{B})$ and $S = \sum_{j=1}^N S_j$.

Proof Applying $\underline{Hom}_{\mathcal{O}_{\mathcal{C}}}(\,\cdot\,, \mathcal{O}_{\mathcal{C}})$ to the exact sequence (1.58), we obtain the exact sequence

$$0 \to \Theta_{\mathcal{C}/\mathcal{B}} \to \Theta_{\mathcal{C}} \to \pi^{-1}\Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}} \to \underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}) \to 0.$$

This exact sequence splits into the following short exact sequences:

$$0 \to \Theta_{\mathcal{C}/\mathcal{B}} \to \Theta_{\mathcal{C}} \xrightarrow{\kappa} \mathcal{M} \to 0. \tag{1.64}$$

$$0 \to \mathcal{M} \to \pi^{-1}\Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}} \to \underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}) \to 0. \tag{1.65}$$

Let $\mathcal{T}$ be the sheaf of germs of holomorphic vector fields on $\mathcal{C}$ preserving n-th infinitesimal neighbourhoods. The sheaf $\mathcal{T}$ is given by

$$\mathcal{T} = \{ \, v \in \Theta_{\mathcal{C}} \, | \, v(\mathcal{I}_S) \subset \mathcal{I}_S \, \}. \tag{1.66}$$

The sheaf $\mathcal{T}$ is an $\mathcal{O}_{\mathcal{C}}$-submodule of $\Theta_{\mathcal{C}}$ and coincides with $\Theta_{\mathcal{C}}$ outside $\bigcup_{j=1}^N S_j$. For a point $P \in S_j$ we let $(u_1, u_2, \ldots, u_M, z)$ be local coordinates of $\mathcal{C}$ with center P such that $(u_1, u_2, \ldots, u_M)$ are the coordinates of $\mathcal{B}$ with center $\pi(P)$ and such that S_j is defined by the equation $z = 0$ in a neighbourhood of P. Then, in a neighbourhood of P the sheaf $\mathcal{T}$ is generated by

$$z\frac{\partial}{\partial z}, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_M}$$

as an $\mathcal{O}_{\mathcal{C}}$-module. Hence $\mathcal{T}$ is locally free on $\mathcal{C}$.

Let us examine the exact sequences (1.64) and (1.65). Since the support of the sheaf $\underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}})$ is in Σ, the sheaf $\mathcal{M}$ is equal to $\pi^{-1}\Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}}$ on $\mathcal{O}_{\mathcal{C}} \setminus \Sigma$. By using the above local coordinates of $\mathcal{C}$ with center $P \in S_j$, the restriction of the mapping κ in (1.64) to $\mathcal{T}$ in a neighbourhood of P is given by

$$a(u,z)z\frac{\partial}{\partial z} + \sum B_j(u,z)\frac{\partial}{\partial u_j} \longmapsto \sum B_j(u,z)\frac{\partial}{\partial u_j}.$$

Hence $\kappa : \mathcal{T} \to \mathcal{M}$ is surjective and its kernel is $\Theta_{\mathcal{C}/\mathcal{B}}(-(n+1)S)$ in a neighbourhood of P. On the other hand, on $\mathcal{B} \setminus \bigcup_{j=1}^{N} S_j$ the sheaf $\mathcal{T}$ is equal to $\Theta_{\mathcal{C}}$. Thus we have an exact sequence

$$0 \to \Theta_{\mathcal{C}/\mathcal{B}}(-S) \to \mathcal{T} \to \mathcal{M} \to 0. \tag{1.67}$$

From the exact sequence (1.67) we obtain the long exact sequence

$$\begin{aligned} 0 \to \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) \xrightarrow{\tau} \pi_*\mathcal{T} \to \pi_*\mathcal{M} \xrightarrow{\rho} R^1\pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) \\ \to R^1\pi_*\mathcal{T} \to R^1\pi_*\mathcal{M} \to 0\,. \end{aligned} \tag{1.68}$$

Put $\mathcal{B}_0 = \mathcal{B} \setminus D$, $\mathcal{C}_0 = \pi^{-1}(\mathcal{B}_0)$, $\pi_0 = \pi|\mathcal{C}_0$. Then on $\mathcal{B}_0$, $\pi_{0*}\mathcal{M} = \Theta_{\mathcal{B}}$ and the homomorphism ρ is the Kodaira-Spencer mapping by Corollary 1.26. Since our family is versal, ρ is an isomorphism on $\mathcal{B}_0$. Therefore, the sheaf homomorphism τ in (1.68) is an isomorphism on $\mathcal{B}_0$. But on $\mathcal{B}_0$ we have $\pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) = 0$. Therefore, $\pi_*\mathcal{T} = 0$ on $\mathcal{B}_0$. As $\mathcal{T}$ is locally free, $\pi_*\mathcal{T}$ is torsion-free, hence $\pi_*\mathcal{T} = 0$ on $\mathcal{B}$. This also implies

$$\pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) = 0 \tag{1.69}$$

on $\mathcal{B}$.

Next we show that ρ in (1.68) is an isomorphism. For that purpose it is enough to show that $R^1\pi_*\mathcal{T}$ is locally free. Because, if $R^1\pi_*\mathcal{T}$ is locally free, as ρ is an isomorphism on $\mathcal{B}_0$, $\operatorname{Coker}\rho$ is a torsion subsheaf of $R^1\pi_*\mathcal{T}$, hence zero. By the cohomology theory of coherent sheaves,

$$\chi(\mathcal{T} \otimes \mathcal{O}_{C_s}) = \dim_{\mathbb{C}} H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) - \dim_{\mathbb{C}} H^1(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s})$$

is independent of $s \in \mathcal{B}$, where $C_s = {\pi^{(n)}}^{-1}(s)$. (See, for example, [BS].) Moreover, if $\dim_{\mathbb{C}} H^1(C_s, \mathcal{T}\otimes\mathcal{O}_{C_s})$ is independent of s, say k, since we have $H^2(C_s, \mathcal{T}\otimes\mathcal{O}_{C_s}) = 0$, $R^1\pi_*\mathcal{T}$ is a locally free $\mathcal{O}_{\mathcal{B}}$-module of rank k on $\mathcal{B}$. Therefore, it is enough to show that $H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) = 0$ for all $s \in \mathcal{B}$.

Since C_s is a locally complete intersection in $\mathcal{C}$, we have an exact sequence

$$0 \to \Theta_{C_s} \to \Theta_{\mathcal{C}} \otimes \mathcal{O}_{C_s} \to \mathcal{O}_{C_s}(N) \to \underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega^1_{\mathcal{O}_{C_s}}, \mathcal{O}_{C_s}) \to 0$$

where N is the normal bundle of C_s in $\mathcal{C}$ which is a trivial bundle of rank $3g-3+N$. (See, for example, [Ar].) From this exact sequence we obtain two short exact sequences

$$0 \to \Theta_{C_s} \to \Theta_{\mathcal{C}} \otimes \mathcal{O}_{C_s} \to M_s \to 0,$$

$$0 \to M_s \to \mathcal{O}_{C_s}(N) \to \underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega_{C_s}, \mathcal{O}_{C_s}) \to 0.$$

Similarly as above we have an exact sequence

$$0 \to \Theta_{C_s}(-\sum_{j=1}^{N} Q_j) \to \mathcal{T} \otimes \mathcal{O}_{C_s} \to M_s \to 0,$$

where $Q_j = s_j(s)$. This gives a long exact sequence

$$\begin{aligned} 0 \quad = \quad & H^0(C_s, \Theta_{C_s}(-\sum Q_j)) \to H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) \to H^0(C_s, M_s) \\ & \xrightarrow{\rho} H^1(C_s, \Theta_{C_s}(-\sum Q_j)) \to H^1(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}). \end{aligned} \tag{1.70}$$

The cohomology group $H^0(C_s, M_s)$ parameterizes infinitesimal displacements of C_s in $\mathcal{C}$. (For the details see Tsuboi [Ts].) Since $\pi : \mathcal{C} \to \mathcal{B}$ is a versal family, infinitesimal displacements of C_s in $\mathcal{C}$ and infinitesimal deformations of C_s coincide. Hence the homomorphism ρ in (1.70) is an isomorphism. Hence we have

$$H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) = 0.$$

Finally we shall show that $\pi_*\mathcal{M}$ is isomorphic to $\Theta_{\mathcal{B}}(-\log D)$. From (1.65) we obtain an exact sequence

$$0 \to \pi_*\mathcal{M} \to \Theta_{\mathcal{B}} \xrightarrow{t} \pi_*(\underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}))\,.$$

The homomorphism t is the same as the one appearing in the exact sequence (1.62). Hence t is surjective. Therefore, by Lemma 1.35 $\pi_*\mathcal{M}$ is isomorphic to $\Theta_{\mathcal{B}}(-\log D)$. □

Remark 1.37 The homomorphism ρ in the above Theorem 1.36 is also called the *Kodaira-Spencer mapping*. The above proof shows that there exists an exact sequence

$$0 \to \Theta_{\mathcal{C}/\mathcal{B}}(-\sum S_j) \to \mathcal{T} \to \pi^{-1}\Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}} \to \underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega^1_{C_s}, \mathcal{O}_{C_s}) \to 0$$

where $\mathcal{T}$ is the subsheaf of $\Theta_{\mathcal{C}}$ defined in (1.66). Choose a small Stein open set $\mathcal{U} \subset \mathcal{B}$ and a vector field $v \in H^0(\mathcal{U}, \Theta_{\mathcal{B}}(-\log D))$. Choose also a Stein open covering $\{\mathcal{U}_j\}_{j\in J}$ of $\pi^{-1}(\mathcal{U})$. Then v also defines an element

$$\pi^*v \in H^0(\mathcal{U}_j, \pi^{-1}\Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}}),$$

whose image in $\underline{Ext}^1_{\mathcal{O}_{C_s}}(\Omega^1_{C_s}, \mathcal{O}_{C_s})$ is zero, since the tangent vector v is a direction of an infinitesimal deformation preserving singularities. Therefore, if $\mathcal{U}_j$ is small enough, we can find an element $v_j \in H^0(\mathcal{U}_j, \mathcal{T})$ which is mapped to π^*v. Then we have

$$v_{ij} = v_j - v_i \in H^0(\mathcal{U}_i \cap \mathcal{U}_j, \Theta_{\mathcal{C}/\mathcal{B}}(-S))$$

and $\{v_{ij}\}$ defines an element

$$[\{v_{ij}\}] \in H^1(\pi^{-1}(\mathcal{U}), \Theta_{\mathcal{C}/\mathcal{B}}(-S)).$$

The mapping

$$v \longmapsto [\{v_{ij}\}]$$

is nothing but the Kodaira-Spencer mapping ρ in Theorem 1.36.

Lemma 1.38 *Let $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}\,; s_1, s_2, \dots, s_N)$ be a versal family of N-pointed* smooth *curves of genus g. Let X, Y be holomorphic vector fields of $\mathcal{B}$ and*

$$\rho : \Theta_{\mathcal{B}} \to R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-S)$$

be the Kodaira-Spencer mapping. Then we have

$$\rho([X,Y]) = [\rho(X), \rho(Y)] + X(\rho(Y)) - Y(\rho(X))$$

where by putting

$$\rho(X) = \{\theta_{\lambda\mu}\}, \qquad \rho(Y) = \{\tau_{\lambda\mu}\}$$

$\rho([X,Y])$ is a cocycle $\{\vartheta_{\lambda\mu}\}$ defined as

$$\vartheta_{\lambda\mu} = [\theta_{\lambda\mu}, \tau_{\lambda\mu}]$$

and $X(\rho(Y))$ is a cocycle $\{\tau'_{\lambda\mu}\}$ defined as

$$\tau'_{\lambda\mu} = X(\tau_{\lambda\mu}).$$

Proof We may assume that $\mathcal{B}$ is small enough. Let $\{U_{\lambda\mu}\}$ be a sufficiently fine open covering of $\mathcal{C}$. Let $(z_1, \ldots, z_m)$ be coordinates of $\mathcal{B}$ and $(z_1, \ldots, z_m, \xi_\lambda)$ be local coordinates of U_λ. Moreover, we may assume that for $j \in \{1, 2, \ldots, N\} \subset \Lambda$, U_j is a coordinate neighbourhood of $s_j(\mathcal{B})$ such that ξ_j induces an n-th infinitesimal neighbourhood η_j and that $s_j(\mathcal{B})$ is defined by $\xi_j = 0$. For $U_\lambda \cap U_\mu \neq \emptyset$, we have

$$\xi_\lambda = h_{\lambda\mu}(\xi_\mu, z_1, \ldots, z_m).$$

The vector field X induces an infinitesimal transformation of the coordinates $(z_1, \ldots, z_m)$:

$$(z_1, \ldots, z_m) \mapsto (z_1 + \epsilon_1 a_1(z), \ldots, z_m + \epsilon_1 a_m(z))$$

where ϵ_1 is a dual number, that is, $\epsilon_1^2 = 0$ and

$$X = \sum_{j=1}^{m} a_i(z) \frac{\partial}{\partial z_i}.$$

For simplicity we express (1.4) as $z \mapsto z + \epsilon_1 X$. Since our family is versal, we have

$$g_{\lambda\mu}(\xi_\mu, z + \epsilon_1 X) = g_{\lambda\mu}(\xi_\mu, z) + \epsilon_1 \ell_{\lambda\mu}(\xi_\mu, z)$$

where we put

$$\begin{aligned} \theta_{\lambda\mu} &= \ell_{\lambda\mu} \frac{d}{d\xi_\mu} \\ \tau_{\lambda\mu} &= m_{\lambda\mu} \frac{d}{d\xi_\mu}. \end{aligned}$$

Similarly we have

$$g_{\lambda\mu}(\xi_\mu, z + \epsilon_2 Y) = g_{\lambda\mu}(\xi_\mu, z) + \epsilon_2 m_{\lambda\mu}(\xi_\mu, z)$$

where ϵ_2 is another dual number. Now let us calculate the result when first we deform infinitesimally in the direction of X then deform infinitesimally in the direction of Y:

$$\begin{aligned} & g_{\lambda\mu}(\xi_\mu, z + \epsilon_2 Y) + \epsilon_1 \ell_{\lambda\mu}(\xi_\lambda + \epsilon_2 m_{\lambda\mu}, z + \epsilon_2 Y) \\ = \quad & g_{\lambda\mu}(\xi_\mu, z) + \epsilon_2 m_{\lambda\mu}(\xi_\lambda, z) + \epsilon_1 \ell_{\lambda\mu}(\xi_\lambda, z) + \epsilon_1\epsilon_2 (Y(\ell_{\lambda\mu}(\xi_\lambda, z)) + m_{\lambda\mu} \frac{d\ell_{\lambda\mu}}{d\xi_\lambda}. \end{aligned}$$

If we deform in the opposite order, we have

$$g_{\lambda\mu}(\xi_\mu, z) + \epsilon_1 \ell_{\lambda\mu}(\xi_\lambda, z) + \epsilon_2 m_{\lambda\mu}(\xi_\lambda, z) + \epsilon_1\epsilon_2 (X(m_{\lambda\mu}(\xi_\lambda, z)) + \ell_{\lambda\mu} \frac{d\ell_{\lambda\mu}}{d\xi_\lambda}.$$

Therefore, if we put

$$\rho([X, Y]) = \{\widetilde{\vartheta}_{\lambda\mu}\}$$

we have

$$\begin{aligned} \widetilde{\vartheta}_{\lambda\mu} &= \{\ell_{\lambda\mu} \frac{dm_{\lambda\mu}}{d\xi_\lambda} - m_{\lambda\mu} \frac{d\ell_{\lambda\mu}}{d\xi_\lambda} + X(m_{\lambda\mu}) - Y(\ell_{\lambda\mu})\} \frac{d}{d\xi_\lambda} \\ &= [\theta_{\lambda\mu}, \tau_{\lambda\mu}] + \{X(\tau_{\lambda\mu}) - Y(\ell_{\lambda\mu}\} \frac{d}{d\xi_\lambda} \\ &= [\rho(X), \rho(Y)] + X(\rho(Y)) - Y(\rho(X)). \end{aligned}$$

□