Preface

Modern topology uses many different methods. In this book, we largely investigate the methods of combinatorial topology and the methods of differential topology; the former reduce studying topological spaces to investigation of their partitions into elementary sets, such as simplices, or covers by some simple sets, while the latter deal with smooth manifolds and smooth maps. Many topological problems can be solved by using any of the two approaches, combinatorial or differential; in such cases, we discuss both of them.

Topology has its historical origins in the work of Riemann; Riemann's investigation was continued by Betti and Poincaré. While studying multivalued analytic functions of a complex variable, Riemann realized that, rather than in the plane, multivalued functions should be considered on two-dimensional surfaces on which they are single-valued. In these considerations, two-dimensional surfaces arise by themselves and are defined intrinsically, independently of their particular embeddings in $\mathbb{R}^3$; they are obtained by gluing together overlapping plane domains. Then, Riemann introduced the notion of what is known as a (multidimensional) manifold (in the German literature, Riemann's term *Mannigfaltigkeit* is used). A manifold of dimension n, or n-manifold, is obtained by gluing together overlapping domains of the space $\mathbb{R}^n$. Later, it was recognized that to describe continuous maps of manifolds, it suffices to know only the structure of the open subsets of these manifolds. This was one of the most important reasons for introducing the notion of topological space; this is a set endowed with a topology, that is, a system of subsets (called open sets) with certain properties.

Chapter 1 considers the simplest topological objects, graphs (one-dimensional complexes). First, we discuss questions which border on geometry, such as planarity, the Euler formula, and Steinitz' theorem. Then, we consider fundamental groups and coverings, whose basic properties are well seen in graphs. This chapter is concluded with a detailed discussion of the polynomial invariants of graphs; there has been much interest in them recently, after the discovery of their relationship with knot invariants.

Chapter 2 is concerned with another fairly simple topological object, Euclidean space with standard topology. Subsets of Euclidean space may have very complicated topological structure; for this reason, only a few basic statements about the topology of Euclidean space and its subsets are included. One of the fundamental problems in topology is the classification of continuous maps between topological spaces (on the spaces certain constraints may be imposed; the classification is up to some equivalence). The simplest classifications of this kind are related to curves in the plane, i.e., maps of S^1 to $\mathbb{R}^2$. First, we prove the Jordan theorem and the Whitney–Graustein classification theorem for smooth closed curves up to regular homotopy. Then, we prove the Brouwer fixed point theorem and Sperner's lemma by several different methods (in addition to the standard statement of Sperner's lemma, we give its refined version, which takes into account the orientations of simplices). We also prove the Kakutani fixed point theorem, which generalizes the theorem of Brouwer. The chapter is concluded by the Tietze theorem on extension of continuous maps, which is derived from Urysohn's lemma, and two theorems of Lebesgue, the open cover theorem, which is used in the rigorous proofs of many theorems from homotopy and homology theories, and the closed cover theorem, on which the definition of topological dimension is based.

Chapter 3 begins with elements of general topology; it gives the minimal necessary information constantly used in algebraic topology. We consider three properties (Hausdorffness, normality, and paracompactness) which substantially facilitate the study of topological spaces. Then, we consider two classes of topological spaces that are most important in algebraic topology (namely, simplicial complexes and CW-complexes), describe techniques for dealing with them (cellular and simplicial approximation), and prove that these spaces have the three properties mentioned above. We also introduce the notion of degree for maps of pseudomanifolds and apply it to prove the Borsuk–Ulam theorem, from which we derive many corollaries. The chapter is concluded with a description of some constructions of topological spaces, including joins, deleted joins, and symmetric products. We apply deleted joins to prove that certain n-dimensional simplicial complexes cannot be embedded in $\mathbb{R}^{2n}$.

Chapter 4 covers very diverse topics, such as two-dimensional surfaces, coverings, local homeomorphisms, graphs on surfaces (including genera of graphs and graph coloring), bundles, and homotopy groups.

Chapter 5 turns to differential topology. We consider smooth manifolds and the application of smooth maps to topology. First, we introduce some basic tools (namely, smooth partitions of unity and Sard's theorem) and consider an example, the Grassmann manifolds, which plays an important role everywhere in topology. Then, we discuss notions related to tangent spaces, namely, vector fields and differential forms. After this, we prove existence theorems for embeddings and immersions (including closed embeddings of noncompact manifolds), which play an important role in the study of smooth manifolds. Moreover, we prove that a closed nonorientable n-manifold cannot be embedded in $\mathbb{R}^{n+1}$ and determine what two-dimensional surfaces can be embedded in $\mathbb{R}P^3$. Further, we introduce a homotopy invariant, the degree of a smooth map, and apply it to define the index of a singular point of a vector field. We prove the Hopf theorem, which gives a homotopy classification of maps $M^n \to S^n$. We also describe a construction of Pontryagin which interprets $\pi_{n+k}(S^n)$ as the set of classes of cobordant framed k-manifolds in $\mathbb{R}^{n+k}$. We conclude this chapter with Morse theory, which relates the topological structure of a manifold to local properties of singular points of a nondegenerate function on this manifold. We give explicit examples of Morse functions on some manifolds, including Grassmann manifolds.

Chapter 6 is devoted to explicit calculations of fundamental groups for some spaces and to applications of fundamental groups. First, we prove a theorem about generators and relations determining the fundamental group of a CW-complex and give some applications of this theorem. Sometimes, it is more convenient to calculate fundamental groups by using exact sequences of bundles. Such is the case for, e.g., the fundamental group of $\mathrm{SO}(n)$. In many situations, the van Kampen theorem about the structure of the fundamental group of a union of two open sets is helpful. For example, it can be used to calculate the fundamental group of a knot complement. At the end of the chapter, we give another theorem of van Kampen, which gives a method for calculating the fundamental group of the complement of an algebraic curve in $\mathbb{C}P^2$. The corresponding calculations for particular curves are fairly complicated; plenty of interesting results have been obtained, but many things are not yet fully understood.

One of the main purposes of this book is to advance in the study of the properties of topological spaces (especially manifolds) as far as possible without employing complicated techniques. This distinguishes it from the majority of topology books.

The book is intended for readers familiar with the basic notions of geometry, linear algebra, and analysis. In particular, some knowledge of open, closed, and compact sets in Euclidean space is assumed.

The book contains many problems, which the reader is invited to think about. They are divided into three groups: (1) *exercises*; solving them should not cause any difficulties, so their solutions are not included; (2) *problems*; they are not so easy, and the solutions to most of them are given at the end of the book; (3) *challenging problems* (marked with an asterisk); each of these problems is the content of a whole scientific paper. They are formulated as problems not to overburden the main text of the book. The solutions to most of these problems are also given at the end of the book. The problems are based on the first- and second-year graduate topology courses taught by the author at the Independent University of Moscow in 2002.

This work was financially supported by the Russian Foundation for Basic Research (project no. 05–01–01012a).