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Preface to the Second Edition

“[When a mathematician speaks about teaching], colleagues smile tolerantly to one another in the same way family members do when grandpa dribbles his soup down his shirt.” Herb Clemens wrote these words in 1988. They were right on point at the time. The amazing fact is that they are no longer true.

Indeed the greatest single achievement of the so-called “teaching reform” movement is that it has enabled, or compelled, all of us to be concerned about teaching. Never mind the shame that in the past we were *not* concerned about teaching. Now we are all concerned, and that is good.

Of course there are differing points of view. The “reform” school of thought favors discovery, cooperative and group learning, use of technology, higher-order skills, and it downplays rote learning and drill. The traditionalists, by contrast, want to continue giving lectures, want the students to do traditional exercises, want the students to take the initiative in the learning process, and want to continue to drill their students.[1] Clearly there are merits in both points of view. The good news is that the two sides are beginning to talk to each other. The evidence? **(1)** A conference held at MSRI in December, 1996 with the sole purpose of helping the two camps to communicate (see the Proceedings in [GKM]); **(2)** The observation that basic skills play a new role, and are positioned in a new way, in the reform curriculum; **(3)** The observation that standard lectures—the stock-in-trade of traditionalists—are not the final word on engaging students in the learning process; **(4)** The fact that studies indicate that neither method is more effective than the other, but that both have strengths; **(5)** The new wave of calculus books (see [STEW]) that attempt a marriage of the two points of view.

The reader of this book may as well know that I am a traditionalist, but one who sees many merits in the reform movement. For one thing, the reform movement has taught us to reassess our traditional methodologies. It has taught us that there is more than one way to get the job done. And it has also taught us something about the sociological infrastructure of twentieth-century mathematics. We see that our greatest pride is also our Achilles heel. In detail, the greatest achievement of twentieth-century mathematics is that we have (to the extent possible) fulfilled the Hilbert/Bourbaki program of putting everything on a rigorous footing; we have axiomatized our subject; we have precise definitions of everything. The bad news is that these accomplishments have shaped our world view, all the way down into the calculus classroom. Because we have taught ourselves to think strictly according to Occam’s Razor, we also think that that should be the mode of discourse in the calculus classroom. This view is perhaps short-sighted.

[1] As I will say elsewhere in the book, the reformers constitute a heterogeneous group, just like the traditionalists. There is no official reform dogma, just as there is no official traditionalist dogma. Some reformers tell me that they strongly favor drill, but that drill should be built atop a bedrock of understanding. Many traditionalists seem to prefer to give the drill first—asking the students to take it on faith—and *then* to develop understanding. George Andrews has asked whether, if instead of calling it “mere rote learning” we called it “essential drill”, would people view it differently? See the Appendix of Andrews at the end of this book.

First, students (and others, too!) do not generally learn *axiomatically* (from the top down). In many instances it is more natural for them to learn *inductively* (from the bottom up). Of course the question of how people learn has occupied educational theorists as far back as Beth and Piaget [BPA], and will continue to do so. But, as I say elsewhere in this book, the mathematics instructor must realize that a student cannot stare at a set of axioms and "see what is going on" in the same way that an experienced mathematician can. Often it is more natural for the student to first latch on to an example.

Second, we must realize that the notion of "proof" is a relative thing. Mathematical facts, or theorems, are freestanding entities. They have a life of their own. But a proof is largely a psychological device for convincing someone that something is true. A trained mathematician is taught a formalism for producing a proof that will be acceptable to his colleagues. But a freshman in college is not. What constitutes a believable proof for a freshman could easily be a good picture, or a plausibility argument. This insight alone can turn an ordinary teacher into a good one. What is the sense in showing a room full of freshmen a perfectly rigorous proof (of the fundamental theorem of calculus, say), secure in the knowledge that you have "done the right thing," but also knowing unconsciously that the students did not understand a word of it? Surely it is more gentle, as a didactic device, to replace "*Proof:*" with "Here is an idea about why this is true." In doing the latter, you have not been dishonest (i.e., you have not claimed that something was a strictly rigorous proof when in fact it was not). You have instead met the students half way. You have spoken to them in their own language. You have appealed to their collective intuition. Perhaps you have taught them something. Always keep in mind that persuasion has many faces.

I have witnessed discussions in which certain individuals were adamant that, if you give an explanation in a calculus class that is not strictly a proof, then you must say, "This is not a proof; it is an informal explanation." Of course such a position is a consequence of twentieth-century mathematical values, and I respect it. But I do not think that it constitutes good teaching. In the first place, such a mantra is both tiresome and discouraging for the students. The instructor can instead say, "Let's think about why this is true ..." or "Here is a picture that shows what is going on ..." and thereby convey the same message in a much friendlier fashion.

In my own mathematics department we have a "transitions" course, in which students are taught first-order logic, naive set theory, equivalence relations and classes, the constructions of the number systems, and the axiomatic method. They are also taught—at a very rudimentary level—how to construct their own proofs. Typically a student takes this course *after* calculus, linear algebra, and ordinary differential equations but *before* abstract algebra and real analysis. I think of the transitions course as a bellwether. Before that course, students are not ready for formal proofs. We should adapt our teaching methodology to *their* argot. After the transitions course, the students are more sophisticated. Now they are ready to learn *our* argot.

I have decided to write this new edition of *How to Teach Mathematics* in part because I have learned a lot about teaching in the five-year interval since this book first appeared. The teaching reform movement has matured, and so have the rest of us. I believe that I now know a lot more about what constitutes good teaching. I regularly teach our graduate student seminar to help prepare our Ph.D. candidates for a career in teaching, and I have an ever better understanding of how to conduct such training. I would like to share my new insights in this edition.

One of the best known mathematical errors, particularly in the study of an optimization problem, is to assume that the problem has a solution. Certainly Riemann's original proof of the Riemann mapping theorem is a dramatic example of this error, but the calculus of variations (for instance) is littered with other examples. Why can we not apply this hard-won knowledge to other aspects of our professional lives? Why do we assume that there is a "best" way to teach calculus? Or a "best" textbook? Teaching is a very personal activity, and different individuals will do it differently. Techniques that work for one person will not work for another. (Also, techniques that work in one class will not necessarily work in another.) I believe that we need, as a group, to acknowledge that there is a pool of worthwhile teaching techniques, and we should each choose those methods that work for us and for our students.

Ever since the first edition of this book appeared, mathematicians have approached me and asked, "OK, what's the secret? Students these days drive me crazy. I can't get through to them. They won't talk to me. How do you do it?" I wish that I had a simple answer. I would like to be able to say, "Take this little green pill." or "Say this prayer in the morning." or "Hold your mouth this way." But in fact there is no simple answer. Even so, I have invested considerable time analyzing the situation as well as talking to other successful teachers about how to make the teaching process work. I have come to the following conclusion.

Students are like dogs: They can smell fear. (I do not mean to say here that we should think of our students as attack dogs. Rather, they are sensitive to body language and to nuances of behavior. See also Section 2.9 on teaching evaluations.) When you walk into your classroom, the students can tell right away whether you really want to be there, whether you have something interesting to tell them, whether you respect them as people. If they sense instead that you are merely slogging through this dreary duty, just writing the theorems and proofs on the blackboard, refusing to answer questions for lack of time, then they will react to you in a correspondingly lackluster manner.

When I walk into my calculus class, I look forward to seeing the students perk up, with a look on their faces that says "Showtime!" In the few minutes before the formal class begins, I chat with them, joke around, find out what is going on in their lives. I relate to them as people. It will never happen that a student will go to the chairman or the dean and complain about me. Why? Because they know that they can come and talk to *me* about their concerns. If a student is not doing well in my class, that student is comfortable coming to me. And he knows that the fault for his poor performance is as likely *his* as it is mine, because he realizes that I am doing everything that I can. If you believe what I am describing here, then perhaps you can also understand why I enjoy teaching, and why I find the process both stimulating and fulfilling.

I recently taught a fairly rigorous course in multivariable calculus—a subject in which students usually have a lot of trouble. The main reason that they have so much trouble is that there are so many ideas—vectors, cross products, elements of surface area, orientation, conservative vector fields, line integrals, tangent planes, etc.—and they are all used together. Just understanding how to calculate both sides of the equation in Stokes's theorem, or the divergence theorem, requires a great deal of machinery. The way that I addressed their difficulties is that I worked the students hard. I gave long, tough homework assignments. A day or two before any given assignment was due, I would begin a class discussion of the homework. If necessary, I would work out the bulk of a problem on the board for them. But I would add that I expected each of them to write up the problem carefully and completely—with full details. And I would give them a few extra days so that they could complete the assignment. But I did not stop there. Next class, I would ask how the homework was going. If necessary, we would discuss it again. If necessary, I would give them another extension. The point here is that I made it absolutely clear to the students that the most important thing to me was that they would complete the assignment. I would give them whatever time, and whatever help, was needed to complete the work. During the long fifteen-week semester, attendance in the class was virtually constant, and always exceeded 95%. At the end, I gave them a long, tough final exam. And the average was 85%. I can only conclude that I set a standard for these students, and they rose to it. Both they and I came away from the course with a feeling of success. They had worked hard, and they had learned something.

You may be thinking, "Well, Krantz teaches at a fancy private school with fancy private students. I could never get away with this at Big State University." That is a defeatist attitude. If you expect your students to try, then you must try. I have taught at big state universities. I understand the limitations that teaching a large class of not particularly select students imposes. But you can adjust the techniques described in the last paragraph to most any situation. If you wonder how I can afford to spend valuable class time going over homework, my answer is this: I am an experienced teacher, and fourteen weeks is a long time. I can always adjust future classes, leave out a few examples, give short shrift to some ancillary topics. I never worry about running out of time.

I have gone on at some length in this Preface to give the uninitiated reader a glimpse of where I am coming from. I hope that on this basis you can decide whether you want to read the remainder of the book. This is a self-help book in the strongest sense of the word. It is a kit that will allow you to build your own teaching methodology and philosophy. I certainly cannot do it for you. What I *can* do is provide you with some tips, and advice, and the benefit of my own experience. Nothing that I say here is "correct" in any absolute sense. It is just what I know.

One of my disappointments pursuant to the first edition of this book is that nobody has taken it as an impetus to write his own book espousing his own teaching philosophy. There have been some reviews of this book—several of them rather strong and critical both of the book and of its author (see [MOO], [BRE]). I welcome such discussions, and would only like to see further discourse. I am delighted to be able to say that several distinguished scholars, who have been active in exploring and discussing teaching issues, have agreed to write

Appendices to this new edition of *How to Teach Mathematics*. Let me stress that these are not all people who agree with me. In fact some of us have had spirited public disagreements. But we all share some common values. We want to discover how best to teach our students. The new Appendices help to balance out the book, and to demonstrate that any teaching question has many valid answers.

When I teach the teaching seminar for our graduate students, the first thing I tell them is this: "In this course, I am *not* going to tell you how to teach. You have to decide that for yourselves. What I intend to do is to sensitize you to certain issues attendant to teaching. Then you will have the equipment so that you can build your own teaching philosophy and style." I would like to suggest that you read this book in the same spirit. You certainly need not agree with everything I say. But I hope you will agree that the issues I discuss are ones that we all must consider as we learn how to teach.

When I was a graduate student—in one of the best math graduate programs in the country—I never heard a single word about teaching. Actually, that's not true. Every once in a while we would be talking about mathematics and someone would look at his watch and say, "Damn! I have to go teach." But that was the extent of it. Six years after I received my Ph.D., I returned to that same Ivy League school as a visiting faculty member. Times had changed, and one of the senior faculty members gave a twenty-minute pep talk to all new instructors. He said, "These days, you can either prove the Riemann hypothesis or you can learn how to teach." He went on to tell us to speak up during lectures, and to write neatly on the blackboard. This was not the most profound advice on teaching that I have ever heard, but it certainly represented progress.

The truth is that, as a graduate student, I was so hellbent on learning to be a mathematician that I probably gave little thought to teaching. I would have felt quite foolish knocking on my thesis advisor's door and asking his advice on how to teach the chain rule. I shudder to think what he might have replied. But we have all evolved. It makes me happy that my own graduate students frequently consult me on **(i)** mathematics, **(ii)** teaching, and **(iii)** the profession. Though I secretly may relish **(i)** a bit more than **(ii)** or **(iii)**, I do enjoy all three.

Teaching is an important part of what we do. Because of economic stringencies, and new societal values, university administrations are monitoring every department on campus to ensure that the teaching is (better than) adequate and is working. My university is known nationwide for its good teaching. Yet an experienced administrator here said recently that 80% of the tenured faculty (campus-wide) could *not* get tenure today on the basis of their teaching.

We simply cannot get away with the carelessness that was our hallmark in the past. Thanks in part to the teaching reform movement, we have all come to understand this change in values, and we are beginning to embrace it. A book like [CAS], which offers advice to a fledgling instructor, could not have existed twenty years ago. Now it is a valuable part of our literature.

Teaching is a regimen that we spend our entire lives learning and revising and honing to a sharp skill. This book is designed to help you in that pursuit.

I am happy to acknowledge the advice and help that I have received from many friends and colleagues in the preparation of this new edition. I would like particularly to mention Joel Brawley, David Bressoud, Robert Burckel, John B. Conway, Ed Dubinsky, Len Gillman, David Hoffman, Gary Jensen, Meyer Jerison, Kristen Lampe, Vladimir Maşek, Chris Mahan, Deborah K. Nelson, Hrvoje Sikic, Nik Weaver, Stephen Zemyan, and Steven Zucker. Lynn Apfel was good enough to read several drafts of this manuscript with painstaking care, and to share with me her cogent insights about teaching; I am most grateful for her contributions. Jennifer Sharp of the American Mathematical Society gave me the benefit both of her editing skills and of her knowledge of language and meaning. Her help has been invaluable.

Last, but not least, Josephine S. Krantz is a constant wellspring of inspiration; her Mom, Randi Ruden, is a source of solace.

Of course the responsibility for all remaining errors or foolishness resides entirely with me.

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Preface to the First Edition

While most mathematics instructors prepare their lectures with care, and endeavor to do a creditable job at teaching, their ultimate effectiveness is shaped by their attitudes. As an instructor ages (and I speak here of myself as much as anyone), he finds that he is less in touch with his students, that a certain ennui has set in, and (alas) perhaps that teaching does not hold the allure and sparkle that it once had. Depending on the sort of department in which he works, he may also feel that hotshot researchers and book writers get all the perks and that "mere teachers" are viewed as drones.

As a result of this fatigue of enthusiasm, a professor will sometimes prepare for a lecture *not* by writing some notes or by browsing through the book but by lounging in the coffee room with his colleagues and bemoaning **(a)** the shortcomings of the students, **(b)** the shortcomings of the text, and **(c)** that professors are overqualified to teach calculus. Fortified by this yoga, the professor will then proceed to his class and give a lecture ranging from dreary to arrogant to boring to calamitous. The self-fulfilling prophecy having been fulfilled, the professor will finally join his cronies for lunch and be debriefed as to **(a)** the shortcomings of the students, **(b)** the shortcomings of the text, and **(c)** that professors are overqualified to teach calculus.

There is nothing new in this. The aging process seems to include a growing feeling that the world is going to hell on a Harley. A college teacher is in continual contact with young people; if he feels ineffectual or alienated as a teacher, then the unhappiness can snowball.

Unfortunately, the sort of tired, disillusioned instructors that I have just described exist in virtually every mathematics department. A college teacher who just doesn't care anymore is a poor role model for the novice instructor. Yet that novice must turn somewhere to learn how to teach. You cannot learn to play the piano or to ski by watching someone else do it. And the fact of having sat in a classroom for most of your life does not mean that you know how to teach.

The purpose of this book is to set down the traditional principles of good teaching in mathematics—as viewed by this author. While perhaps most experienced mathematics instructors would agree with much of what is in this book, in the final analysis this tract must be viewed as a personal polemic on how to teach.

Teaching is important. University administrations, from the top down, are today holding professors accountable for their teaching. Both in tenure and promotion decisions and in the hiring of new faculty, mathematics (and other) departments must make a case that the candidate is a capable and talented teacher. In some departments at Harvard, a job candidate must now present a "teaching dossier" as well as an academic dossier. It actually happens that good mathematicians who are really rotten teachers do not get that promotion or do not get tenure or do not get the job that they seek.

The good news is that it requires no more effort, no more preparation, and no more time to be a good teacher than to be a bad teacher. The proof is in this book. Put in other words, this book is not written by a true believer who is going to exhort you to dedicate every waking hour to learning your students' names and designing seating charts. On the contrary, this book is written by a pragmatist who values his time and his professional reputation, but is also considered to be rather a good teacher.

I intend this book primarily for the graduate student or novice instructor preparing to sally forth into the teaching world; but it also may be of some interest to those who have been teaching for a few or even for several years. As with any endeavor that is worth doing well, teaching is one that will improve if it is subjected to periodic re-examination.

Let me begin by drawing a simple analogy: By the time you are a functioning adult in society, the basic rules of etiquette are second nature to you. You know instinctively that to slam a door in someone's face is **(i)** rude, **(ii)** liable to invoke reprisals, and **(iii)** not likely to lead to the making of friends and the influencing of people. The keys to good teaching are at approximately the same level of obviousness and simplicity. But here is where the parallel stops. We are all *taught* (by our parents) the rules of behavior when we are children. Traditionally, we (as mathematicians) are not taught anything, when we are undergraduate or graduate students, about what constitutes sound teaching.

In the past we have assumed that either

(i) Teaching is unimportant.

or

(ii) The components of good teaching are obvious.

or

(iii) The budding professor has spent a lifetime sitting in front of professors and observing teaching, both good and bad; surely, therefore, this person has made inferences about what traits define an effective teacher.

I have already made a case that **(i)** is false. I agree wholeheartedly with **(ii)**. The rub is **(iii)**. If proof is required that at least some mathematicians have given little thought to exposition and to teaching, then think of the last several colloquia that you have heard. How many were good? How many were inspiring? This is supposed to be the stuff that matters—getting up in front of our peers and touting our theorems. Why is it that people who have been doing it for twenty or thirty years still cannot get it right? Again, the crux is item **(iii)** above. There are some things that we do not learn by osmosis. How to lecture and how to teach are among these.

Of course the issue that I am describing is not black and white. If there were tremendous peer support in graduate school and in the professorial ranks for great teaching, then we would force ourselves to figure out how to teach well. But often there is not. The way to make points in graduate school is to ace the qualifying exams and then to write an excellent thesis. It is unlikely that your thesis advisor wants to spend a lot of time with you chatting about how to teach the chain rule. After all, he has tenure and is probably more worried about where his next theorem or next grant or next raise is coming from than about such prosaic matters as calculus.

The purpose of this book is to prove that good teaching requires relatively little effort (when compared with the alternative), will make the teaching process a positive part of your life, and can earn you the respect of your colleagues. In large part I will be stating the obvious to people who, in theory, already know what I am about to say.

It is possible to argue that we are all wonderful teachers, simply by *fiat*, but that the students are too dumb to appreciate us. Saying this, or thinking it, is analogous to proposing to reduce crime in the streets by widening the sidewalks. It is doubletalk. If you are not transmitting knowledge, then you are not teaching. We are not hired to train the ideal platonic student. We are hired to train the particular students who attend our particular universities. It is our duty to learn how to do so.

This is a rather personal document. After all, teaching is a rather personal activity. But I am not going to advise you to tell jokes in your classes, or to tell anecdotes about mathematicians, or to dress like Gottfried Wilhelm von Leibniz when you teach the product rule. Many of these techniques only work for certain individuals, and only in a form suited to those individuals. Instead I wish to distill out, in this book, some universal truths about the teaching of mathematics. I also want to go beyond the platitudes that you will find in books about teaching *all* subjects (such as "type all your exams", "grade on a bell-shaped curve") and talk about issues that arise specifically in the teaching of mathematics. I want to talk about principles of teaching that will be valid for all of us.

My examples are drawn from the teaching of courses ranging from calculus to real analysis and beyond. Lower-division courses seem to be an ideal crucible in which to forge teaching skills, and I will spend most of my time commenting on those. Upper-division courses offer problems of their own, and I will say a few words about those. Graduate courses are dessert. You figure out how you want to teach your graduate courses.

There are certainly differences, and different issues, involved in teaching every different course; the points to be made in this book will tend to transcend the seams and variations among different courses. If you do not agree in every detail with what I say, then I hope that at least my remarks will give you pause for thought. In the end, you must decide for yourself what will take place in your classroom.

There is a great deal of discussion these days about developing new ways to teach mathematics. I'm all for it. So is our government, which is generously funding many "teaching reform" projects. However, the jury is still out regarding which of these new methods will prove to be of lasting value. It is not clear yet exactly how `Mathematica` notebooks or computer algebra systems or interactive computer simulations should be used in the lower-division mathematics classroom. Given that a large number of students need to master a substantial amount of calculus during the freshman year, and given the limitations on our resources, I wonder whether alternatives to the traditional lecture system—such as, for instance, Socratic dialogue—are the correct method for getting the material across. Every good new teaching idea should be tried. Perhaps in twenty years some really valuable new techniques will have evolved. They do not seem to have evolved yet.

In 1993 I must write about methods that I know and that I have found to be effective. Bear this in mind: Experimental classes are experimental. They usually lie outside the regular curriculum. It will be years before we know for sure whether students taught with the new techniques are understanding and retaining the material satisfactorily and are going on to successfully complete their training. Were I to write about some of the experimentation currently being performed then this book would of necessity be tentative and inconclusive.

There are those who will criticize this book for being reactionary. I welcome their remarks. I have taught successfully, using these methods, for twenty years. Using critical self examination, I find that my teaching gets better and better, my students appreciate it more, and (most importantly) it is more and more effective. I cannot in good conscience write of unproven methods that are still being developed and that have not stood the test of time. I leave that task for the advocates of those methods.

In fact I intend this book to be rather prescriptive. The techniques that I discuss here are ones that have been used for a long time. They work. Picasso's revolutionary techniques in painting were based on a solid classical foundation. By analogy, I think that before you consider new teaching techniques you should acquaint yourself with the traditional ones. Spending an hour or two with this book will enable you to do so.

I am grateful to the Fund for the Improvement of Post-Secondary Education for support during a part of the writing of this book. Randi D. Ruden read much of the manuscript critically and made decisive contributions to the clarity and precision of many passages. Josephine S. Krantz served as a valuable assistant in this process. Bruce Reznick generously allowed me to borrow some of the ideas from his book *Chalking It Up.* I also thank Dick Askey, Brian Blank, Bettye Anne Case, Joe Cima, John Ewing, Mark Feldman, Jerry Folland, Ron Freiwald, Paul Halmos, Gary Jensen, John McCarthy, Alec Norton, Mark Pinsky, Bruce Reznick, Richard Rochberg, Bill Thurston, and the students in our teaching seminar at Washington University for many incisive remarks on different versions of the manuscript. The publications committees of the Mathematics Association of America and of the American Mathematical Society have provided me with detailed reviews and valuable advice for the preparation of the final version of this book.

SGK