
Preface

Have you ever had problems with your shoelaces? With broken shoelaces? With shoelaces that constantly come undone? Tripped over your shoelaces? No, that is not what I mean. What I have in mind are mathematical problems such as the following:

- What is the shortest way to lace your shoes?
- What is the strongest way to lace your shoes?
- How many ways are there to lace your shoes?

On December 5, 2002, a short article of mine [21] that addresses these questions appeared in the journal *Nature*. I don't think anybody was more surprised than I by the incredible amount of publicity attracted by this note on as innocent a topic as shoelaces. In the weeks following its publication, the article was reported on by virtually every major newspaper worldwide, and I received close to one thousand e-mails in which people from all walks of life asked me about the mathematics of shoelaces. This is even more remarkable since the article was not even one page long and merely contained a summary of some of my answers to the above questions without any proofs. This set of notes has been compiled in an attempt to provide the comprehensive account of shoelace mathematics that many people have asked me for.

To start with, pondering questions about the mathematics of shoelaces was not much more than idle doodling on my part. It soon became clear to me that other mathematicians had already thought about the shortest shoelace problem and had come up with some very complete and neat theorems, arrived at via conceptually appealing proofs; see [13], [14], [16], and [20]. This initial trend towards beautiful results continued throughout my subsequent investigations, and, in the end, a very complete picture emerged, consisting mainly of simple, beautiful, and often surprising characterizations of the most common shoelace patterns, arrived at via elementary, yet pretty and nonobvious, mathematics. I think such a picture is worth painting in detail, as many mathematically minded people will be interested in it for as long as they use shoelaces to tie their shoes.

Summary of Contents

In Chapter 1, we collect the most basic definitions and results about n -lacings of a mathematical shoe, the mathematical counterparts of lacings of a shoe with n pairs

of eyelets. Here, a mathematical shoe consists of $2n$ eyelets which are the points of intersection of two vertical lines in the plane that are one unit apart and n equally spaced horizontal lines that are h units apart. An n -lacing of a mathematical shoe is a closed path in the plane consisting of $2n$ line segments whose endpoints are the $2n$ eyelets of the shoe. Furthermore, we require that given any eyelet E , at least one of the two segments ending in it is not contained in the same column of eyelets as E . This condition ensures that every eyelet genuinely contributes towards pulling the two sides of the shoe together or, less formally, that a lacing doesn't have "gaps".

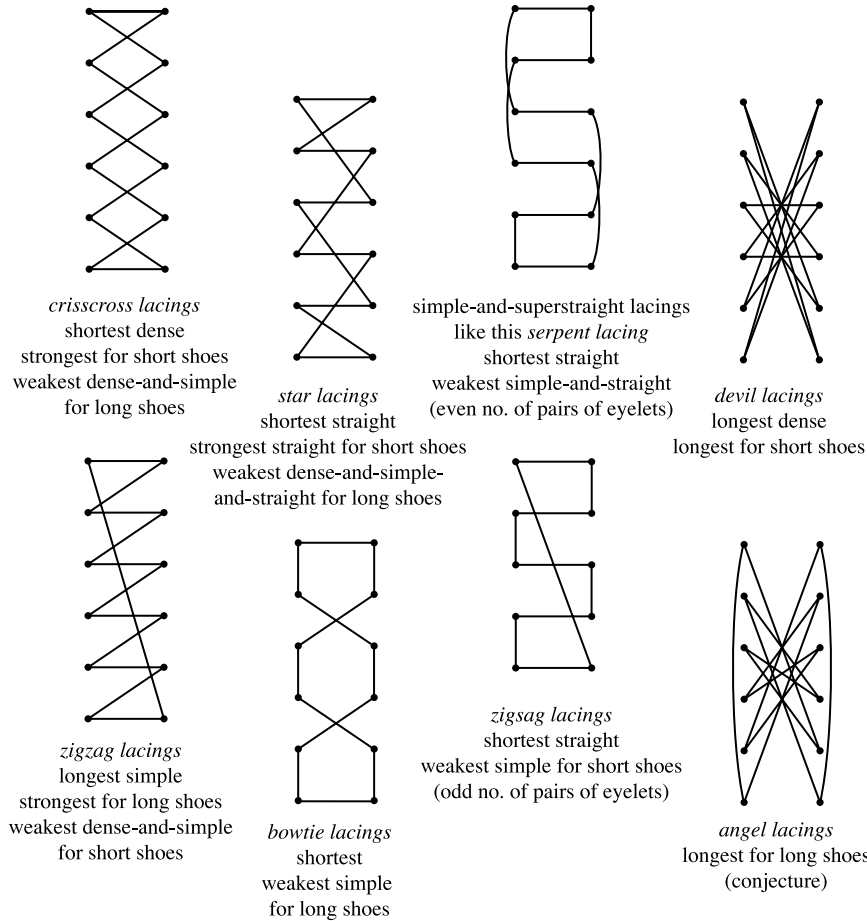
We introduce four important special classes of n -lacings. The *dense* n -lacings are the n -lacings in which the shoelace zigzags back and forth between the two columns of eyelets. The *straight* n -lacings are those n -lacings that contain all possible horizontal segments. The *superstraight* n -lacings are the straight n -lacings all of whose nonhorizontal segments are verticals. Finally, if, when you trace an n -lacing, you move exactly once from the top to the bottom of the shoe and once from the bottom to the top and if you neither "backtrack" on the way down nor on the way up, then the n -lacing is called *simple*.

We describe some families of n -lacings, representatives of which are actually used for lacing real shoes and which pop up in the different characterizations that this set of notes is all about. See the diagram on the next page for a quick visual description of these and other important families of n -lacings and a summary of the most important such characterizations. For example, the two most commonly used n -lacings are the so-called crisscross and zigzag n -lacings, which are featured on the left side of the diagram. As you can see, they both have very neat extremal properties.

In Chapter 2, we consider one-column n -lacings. Imagine pulling really hard on the two ends of the shoelace in one of your shoes that has been laced using a straight lacing. Then, if the lacing does not get in the way and if your foot is narrow enough, you will end up with the two columns of eyelets superimposed, one on top of the other. This means that we do not have to distinguish any longer between the two columns of eyelets and what we are dealing with is a one-column n -lacing, that is, a lacing of just one column of n eyelets in which every eyelet gets visited exactly once. We identify the shortest and longest one-column n -lacings and find the numbers of such lacings. In a final section, we describe a simple method that allows us to construct all straight n -lacings that contract to a given one-column n -lacing. This method plays an important role in deriving the shortest and longest straight n -lacings in subsequent chapters.

In Chapter 3, we derive one formula each for the numbers of those n -lacings that belong to one of ten different classes of n -lacings considered by us: general, dense, simple, straight, dense-and-simple, dense-and-straight, etc. The highlights of this chapter are the formula for the number of n -lacings and the formula for the number of simple n -lacings. Especially, the latter is a very striking example of a simple mathematical object giving rise to a beautiful, yet surprisingly complicated, formula. Also included in this chapter are complete lists of all 2-lacings, all 3-lacings, and all simple 4-lacings.

In Chapter 4, we extend results by Halton [13] and Isaksen [16] by deriving the shortest n -lacings in the different classes of n -lacings in which we are interested. Highlights of this chapter are our proofs that the bowtie n -lacings are the shortest n -lacings overall, that the crisscross n -lacings are the shortest dense n -lacings, and



that the star n -lacings are the shortest among all those n -lacings that are both dense and straight.

In Chapter 5, we consider variations of the shortest shoelace problem. For example, we derive the solution of the shortest shoelace problem for lacings that are not closed, for lacings consisting of more than one shoelace, and for lacings of shoes in which the eyelets are not perfectly aligned. In particular, generalizing a result by Misiurewicz [20], we show that the crisscross n -lacing of a general n -shoe is the shortest dense n -lacing of this shoe. This last result demonstrates that the solution of the shortest shoelace problem in the class of dense n -lacings is very robust, as it stays unchanged even when the underlying array of eyelets is perturbed quite radically. On the other hand, we also show that the same cannot be said about the bowtie n -lacings as the shortest n -lacings overall.

In Chapter 6, we derive the longest n -lacings in most of the different classes of n -lacings in which we are interested. One of the most surprising and interesting results of this chapter is the characterization of the zigzag n -lacings as the longest simple n -lacings.

In Chapter 7, we consider n -lacings as pulley systems and succeed in identifying the strongest pulleys in all the different classes of n -lacings in which we are inter-

ested. Most importantly, we prove that the crisscross and zigzag n -lacings are the strongest pulley systems among all n -lacings and that the star and zigzag n -lacings are the strongest straight n -lacings.

In Chapter 8, we derive the weakest n -lacings in some of the classes of n -lacings in which we are interested. This results in a number of new characterizations of the bowtie, crisscross, zigzag, zigsag, and star n -lacings.

In Appendix A, we first give a brief introduction to the so-called traveling salesman problems. These problems are close relatives of our shortest shoelace problems. We also describe the so-called shoelace formula for calculating the area of polygons.

In Appendix B, we collect all kinds of curious and interesting facts about real shoelaces and lacings.

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