
Preface

In February of 2007, I converted my “What’s new” web page of research updates into a blog at `terrytao.wordpress.com`. This blog has since grown and evolved to cover a wide variety of mathematical topics, ranging from my own research updates, to lectures and guest posts by other mathematicians, to open problems, to class lecture notes, to expository articles at both basic and advanced levels.

With the encouragement of my blog readers, and also of the AMS, I published many of the mathematical articles from the first year (2007) of the blog as [Ta2008b], which will henceforth be referred to as *Structure and Randomness* throughout this book. This gave me the opportunity to improve and update these articles to a publishable (and citeable) standard, and also to record some of the substantive feedback I had received on these articles by the readers of the blog. Given the success of the blog experiment so far, I am now doing the same for the second year (2008) of articles from the blog. This year, the amount of material is large enough that the blog will be published in two volumes.

As with *Structure and Randomness*, each part begins with a collection of expository articles, ranging in level from completely elementary logic puzzles to remarks on recent research, which are only loosely related to each other and to the rest of the book. However, in contrast to the previous book, the bulk of these volumes is dominated by the lecture notes for two graduate courses I gave during the year. The two courses stemmed from two very different but fundamental contributions to mathematics by Henri Poincaré, which explains the title of the book.

This is the second of the two volumes, and it focuses on geometry, topology, and partial differential equations. In particular, Chapter 2 contains

the lecture notes for my course on the famous *Poincaré conjecture* that every simply connected compact three-dimensional manifold is homeomorphic to a sphere, and its recent spectacular solution [Pe2002], [Pe2003], [Pe2003b] by Perelman. This conjecture is purely topological in nature, and yet Perelman's proof uses remarkably little topology, instead working almost entirely in the realm of Riemannian geometry and partial differential equations, and specifically in a detailed analysis of solutions to Ricci flows on three-dimensional manifolds, and the singularities formed by these flows. As such, the course will incorporate, along the way, a review of many of the basic concepts and results from Riemannian geometry (and to a lesser extent, from parabolic PDE), while being focused primarily on the single objective of proving the Poincaré conjecture. Due to the complexity and technical intricacy of the argument, we will not be providing a fully complete proof of this conjecture here (see [MoTi2007] for a careful and detailed treatment); but we will be able to cover the high-level features of the argument, as well as many of the specific components of that argument, in full detail, and the remaining components are sketched and motivated, with references to more complete arguments given. In principle, the course material is sufficiently self-contained that prior exposure to Riemannian geometry, PDE, or topology at the graduate level is not strictly necessary, but in practice, one would probably need some comfort with at least one of these three areas in order to not be totally overwhelmed by the material. (I ran this course as a topics course; in particular, I did not assign homework.)

A remark on notation

For reasons of space, we will not be able to define every single mathematical term that we use in this book. If a term is italicised for reasons other than emphasis or definition, then it denotes a standard mathematical object, result, or concept, which can be easily looked up in any number of references. (In the blog version of the book, many of these terms were linked to their Wikipedia pages, or other on-line reference pages.)

I will however mention a few notational conventions that I will use throughout. The cardinality of a finite set E will be denoted $|E|$. We will use the asymptotic notation $X = O(Y)$, $X \ll Y$, or $Y \gg X$ to denote the estimate $|X| \leq CY$ for some absolute constant $C > 0$. In some cases we will need this constant C to depend on a parameter (e.g. d), in which case we shall indicate this dependence by subscripts, e.g. $X = O_d(Y)$ or $X \ll_d Y$. We also sometimes use $X \sim Y$ as a synonym for $X \ll Y \ll X$.

In many situations there will be a large parameter n that goes off to infinity. When that occurs, we also use the notation $o_{n \rightarrow \infty}(X)$ or simply $o(X)$ to denote any quantity bounded in magnitude by $c(n)X$, where $c(n)$

is a function depending only on n that goes to zero as n goes to infinity. If we need $c(n)$ to depend on another parameter, e.g. d , we indicate this by further subscripts, e.g. $o_{n \rightarrow \infty; d}(X)$.

We will occasionally use the averaging notation

$$\mathbf{E}_{x \in X} f(x) := \frac{1}{|X|} \sum_{x \in X} f(x)$$

to denote the average value of a function $f : X \rightarrow \mathbf{C}$ on a non-empty finite set X .

Acknowledgments

The author is supported by a grant from the MacArthur Foundation, by NSF grant DMS-0649473, and by the NSF Waterman award. The author is also greatly indebted to Peter Petersen and William Wylie for many useful discussions on Ricci flow, which were instrumental in the preparation of the lecture notes on this topic.