Foreword

A well-written book invites the reader into the mind of the author. Novels, stories, books of poetry do this on an unconscious level, sweeping the reader along with the words. A book of mathematics does it differently: well-written mathematics impels the reader to take a pencil in hand and have paper at the ready. The present volume is offered as an enhancement to just such a reading of Jacques Hadamard: *Lessons in Geometry, I. Plane Geometry*, by the American Mathematical Society, Providence, RI, and Education Development Center, Inc., Newton, MA, 2008.

Jacques Hadamard was among the greatest mathematicians of the twentieth century. He made signal contributions to a number of fields, including number theory, differential geometry, and differential equations. But his mind could not be confined to the upper reaches of mathematical thought. His legacy includes a book[1] in which he reflects on the process of creating mathematics, both his own and others. He was active in the Dreyfuss Affair (his wife was related to Alfred Dreyfuss), and held and expressed strong political and philosophical views all his life.[2]

And he was a teacher. For several years, as a graduate student, he worked in a *lycée*, teaching elementary mathematics. Later, the mathematician Gaston Darboux, involved in rewriting the French school mathematics program, thought of Hadamard's experience, and asked him to write a book for teachers on elementary synthetic geometry. The result was a massive two-volume work on plane and solid geometry. The present book is a reader's companion to the translation of the first volume of this work referred to above.

As might be expected of a great mathematician, Hadamard was a master poser of problems. Although termed "exercises", the problems in his *Geometry* are an integral part of the plan of the book. Indeed, the text can be read as a minimal exposition, providing the mathematics that will support the solution of the ensuing problems.

That is, the problems interpret the text, in the way that the harmony interprets the melody in a well-composed piece of music.

The problems are rich and complex. Some of them embroider the text, digging deep into the intuitions behind Hadamard's theorems and lemmas. Others, such as the problems about the Simson line or the nine-point circle, extend the text in various directions. Often these give results which are important in their own right. Very few of them are in any way routine: rarely will the solver read the problem and know immediately how to approach it.

[1] *The Psychology of Invention in the Mathematical Field*, Princeton University Press, 1945.

[2] For many more interesting details of Hadamard's life, see Vladimir Mazya, Tatyana Shaposhnikova, *Jacques Hadamard, A Universal Mathematician*, American Mathematical Society, Providence, Rhode Island, 1998.

For all these reasons, this companion volume to Hadamard's *Geometry* can add significantly to the reader's experience. It requires of the reader only the background of high school geometry, which the text itself provides. The solutions strive to connect the general methods given in the text with intuitions that are natural to the subject, giving as much motivation as possible as well as rigorous and formal solutions. Ideas for further exploration are often suggested.

Another aspect of this companion volume is pedagogical. The work provides indications of possible motivation for difficult turns in the argument, all drawn from classroom experience. Software explorations (using dynamic geometry software) are suggested throughout. (The text of *Geometry* comes with a sample of such explorations, implemented on the TI-Nspire™ Learning Software.) While directly addressed to high school teachers, this style of exposition will speak to any sophisticated lover of classical synthetic geometry, whatever her or his professional interest.

A few notes about the form of the solutions are in order. Hadamard made a judicious choice of material for his text, but omitted certain results that come up often (and are usually easy to prove) in working the problems. These are stated here as lemmas. This is not the usual use of this term: a "lemma" is more often a special case of a more general result to come, or a result about a very specific situation which will not arise later. The lemmas here are separated out from the problems, usually, because they are of interest in their own right. For the same reason, a lemma in one problem is sometimes referenced in another problem.

Other references are to the text of Hadamard's *Geometry* itself. These appear in the form "see **27**", where **27** refers to article (*not* page) 27 in *Geometry.* For reasons of space, this volume omits solutions to Exercises 343 to 422 in Hadamard's original text. These "miscellaneous problems" (Hadamard's own term) are some of the most interesting and complicated in the book. Solutions to these problems can be found on the internet at www.ams.org/bookpages/mbk-70. In some cases, reference is made in the present volume to solutions to these exercises.

It takes a village to write a book like this, and I welcome this occasion to express my gratitude to many people who contributed to it, either by suggesting solutions, or reviewing the content, or assisting with the logistics of getting the manuscript out the door. Vincent Matsko, Don Barry, Yvonne Lai, Alon Amit, Sergei Markelov, and Jordan Tabov all read through the manuscript and corrected some egregious errors in the first versions. Al Cuoco, Jim Sotiros, Tatyana Shubin, Valeriy Ryzhek, Alexander Shen, Borislav Lazarov, and Jenny Sendova all pitched in at those awkward moments when something needed to be done that I couldn't do myself. Sergei Gelfand at the American Mathematical Society was at once patient and prodding in his support of my efforts, urging me on just when my energy flagged. Jennifer Wright Sharp and Gil Poulin, also at AMS, spent hours on the tedious but vital tasks of copy editing, catching errors, omissions, and inconsistencies that the author's jaded eye could not find.

A special mention must be made of Alexei Kopylov, whose detailed and thoughtful reading of most of the manuscript turned up errors large and small, whose suggestions smoothed over rough patches in the exposition, and whose ideas for alternatives enriched the presentation. The work bears the stamp of his careful, detailed, and inspired work.

But perhaps the most important credit goes to the late Dmitri Ivanovitch Perepelkin, a member of the mathematics faculty at Moscow State University from the mid-1930s to his death in 1954. A first-rate geometer, he prepared, in several editions, a translation of Hadamard's *Geometry* into Russian, and supplied solutions to the problems. His solutions were written for an audience of considerable mathematical sophistication. I have borrowed heavily and shamelessly from his work. Many of the solutions—and most of the really clever solutions—are based on his. However, they are reworked significantly, in the hope of making them accessible to a wider audience.

Of course, I hold myself responsible for any incompleteness or errors that remain despite the help of Perepelkin and all the others I have mentioned. A list of errata for Hadamard's *Geometry* appears at http://www.ams.org/bookpages/mbk-57. A list of errata for this volume appears at http://www.ams.org/bookpages/mbk-70.

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On a personal level, I would like to thank my high school French teachers, Ann Bosch, Louis Fuhrman, Josephine Burstein, and Dorothy Roth, for giving me access to that language. I thank my grandfather, Froim Camenir, for transmitting to me a love of language and learning, and helping me in my faltering first steps in learning both mathematics and Russian. And I must thank my wife, Carol Saul, whose support means more to me than is appropriate to express in this context.

I drew on the support of all these people in completing this project. One of the greatest difficulties was the richness of the problems themselves. I found myself thinking for days on some of them, examining and re-examining solutions, extensions, and generalizations, as the book did its work of drawing me into the mind of the author himself. At some point—actually at many points—I had to force myself to relinquish the problem, and remind myself that my role is not to keep them for my own amusement or edification, but to offer them to the mathematical community.

I do so in the hope that readers will enjoy going further than I have in exploring these problems.

Mark Saul
August 2009

CHAPTER 1

Solutions and Comments for Problems in Book I

Exercise 1. *Given a segment AB and its midpoint M, show that the distance CM is one half the difference between CA and CB if C is a point on the segment. If C is on line AB, but not between A and B, then CM is one half the sum of CA and CB.*

Solution. If point C lies between M and B (Figure t1a), we have $AM = AC - MC$ and $MB = MC + CB$. Since $AM = MB$, we have $AC - MC = MC + CB$, so that $MC = \frac{1}{2}(AC - BC)$. A similar argument holds if C lies between A and M.

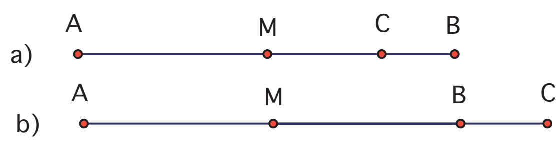

FIGURE t1

If point C lies on the extension of segment AB past point B (Figure t1b), then $AM = MB$ leads to $AC - MC = MC - BC$, so that $MC = \frac{1}{2}(AC + BC)$, and similarly if point C lies on the extension of segment AB past point A.

Notes. It is a useful exercise for students, using dynamic geometry software, to compute the measure of $AC - BC$ and $AC + BC$ as point C slides along line AB. The result—that the sum is constant inside the segment while the difference is constant (in absolute value) outside the segment—is demonstrated dramatically.

Exercise 2. *Given an angle $\widehat{AOB}$ and its bisector OM, show that angle $\widehat{COM}$ is one half the difference of $\widehat{COA}$ and $\widehat{COB}$ if ray OC is inside angle $\widehat{AOB}$; it is the supplement of half the difference if ray OC is inside $\widehat{A'OB'}$, which is vertical to $\widehat{AOB}$; and it is one half the sum of $\widehat{COA}$ and $\widehat{COB}$ if OM is inside one of the other angles $\widehat{AOA'}$ and $\widehat{BOB'}$ formed by these lines.*

Solution. If ray OC_1 lies inside $\widehat{MOB}$ (Figure t2), then $\widehat{AOM} = \widehat{MOB} = \widehat{C_1OA} - \widehat{C_1OM} = \widehat{C_1OB} + \widehat{C_1OM}$, so that $\widehat{C_1OM} = \frac{1}{2}(\widehat{C_1OA} - \widehat{C_1OB})$, with a similar result if ray OC_1 lies inside $\widehat{AOM}$.

If ray OC_2 lies inside $\widehat{A'OB'}$, then its extension OC_1 lies inside $\widehat{AOB}$, and $\widehat{MOC_2} = 180° - \widehat{MOC_1}$, and the required result follows.

If ray OC_3 (not shown in diagram) lies inside $\widehat{BOA'}$, then $\widehat{AOC_3} - \widehat{MOC_3} = \widehat{MOC_3} - \widehat{BOC_3}$, so that $\widehat{MOC_3} = \frac{1}{2}(\widehat{AOC_3} + \widehat{BOC_3})$, and similarly if ray OC_3 lies inside $\widehat{AOB'}$.

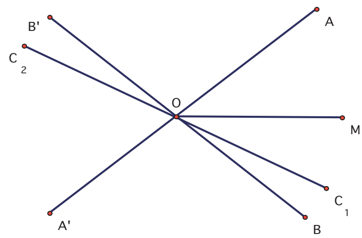

FIGURE t2

Notes. A task analogous to that described in the note to Exercise 1 can be interesting. It is useful to assign students only the first half of this exercise (without the note about the angle vertical to $\widehat{AOB}$). They usually don't think of the rest of the result, but it comes up, again dramatically, if they experiment with dynamic geometry software. As they rotate ray OC_1 around point O, they will find a second *patch* of constant sums. They can guess, then prove, that this patch lies within the vertical angle. (Some software packages may measure angles slightly differently from others. This result holds no matter how the software measures angles, although the value of the constant may differ.)

Exercise 3. *Four rays OA, OB, OC, OD issue from O (in the order listed) such that $\widehat{AOB} = \widehat{COD}$ and $\widehat{BOC} = \widehat{DOA}$. Show that OA and OC are collinear, as are OB and OD.*

Solution. It follows from the problem statement that $\widehat{AOB}+\widehat{BOC} = \widehat{COD}+\widehat{DOA}$ (Figure t3). But clearly $\widehat{AOB} + \widehat{BOC} + \widehat{COD} + \widehat{DOA} = 360°$. Therefore, $\widehat{AOB} + \widehat{BOC} = \widehat{COD} + \widehat{DOA} = 180°$. This last equality implies (by the converse theorem of **15**) that OA and OC are extensions of each other, so B, O, and D are collinear. Similarly, A, O, and C are collinear.

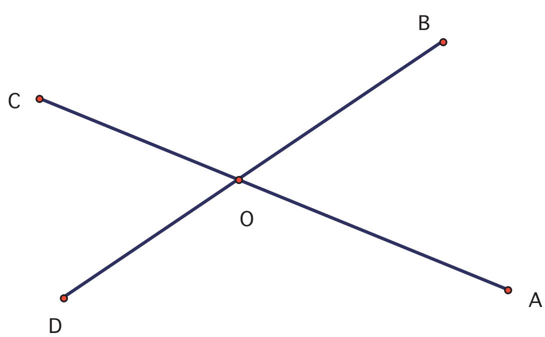

FIGURE t3

Notes. This exercise provides a converse to the theorem of **12**. This theorem can be phrased as: *If two lines intersect, forming four rays meeting at a point, then the opposite angles around that point are equal.* The converse then states: *If* *opposite angles formed by four rays meeting at a point are equal, then the four rays form two lines.*

Exercise 4. *If four consecutive rays OA, OB, OC, OD are such that the bisectors of angles $\widehat{AOB}$, $\widehat{COD}$ are collinear, as are the bisectors of $\widehat{BOC}$, $\widehat{AOD}$, then these rays are collinear in pairs.*

Solution. We have (Figure t4):

$$\frac{1}{2}\widehat{AOB} + \widehat{BOC} + \frac{1}{2}\widehat{COD} = 180^\circ,$$
$$\frac{1}{2}\widehat{COD} + \widehat{DOA} + \frac{1}{2}\widehat{AOB} = 180^\circ.$$

It follows that $\widehat{BOC} = \widehat{DOA}$. Similarly we can show that $\widehat{AOB} = \widehat{COD}$. The result of Exercise 3 (above) then shows that the rays are collinear in pairs.

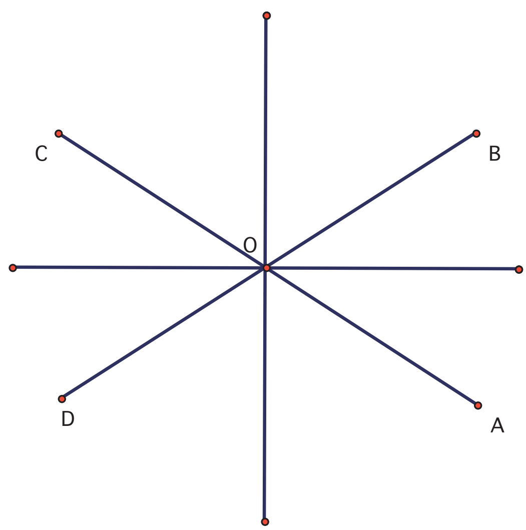

FIGURE t4

Notes. This exercise provides a converse to the result of **17**.

Exercise 5. *Prove that a triangle is isosceles in the following cases*:
1°. *if an angle bisector is also an altitude;*
2°. *if a median is also an altitude;*
3°. *if an angle bisector is also a median.*

Solutions. 1°. Suppose that in triangle ABC we have $\widehat{BAD} = \widehat{CAD}$ and $\widehat{BDA} = \widehat{CDA} = 90^\circ$ (Figure t5). Then triangles ABD, ACD are congruent (ASA, **24**) since they have side AD in common and equal corresponding angles at vertex A and vertex D. It follows from these congruent triangles that $AB = AC$.

2°. If $\widehat{BDA} = \widehat{CDA} = 90^\circ$ and $BD = DC$, then triangles ABD and ACD are congruent (SAS, **24**) , so $AB = AC$.

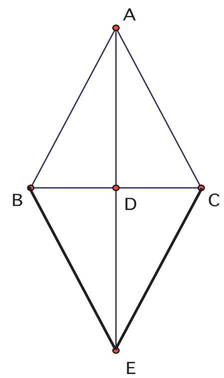

FIGURE t5

3°. Suppose $\widehat{BAD} = \widehat{CAD}$ and $BD = CD$. We extend AD to E so that $AD = DE$ (Figure t5). Triangles ABD and ECD are congruent (by SAS), so $AB = EC$ and $\widehat{BAD} = \widehat{CED}$. Then $\widehat{BAD} = \widehat{CAD}$ and $\widehat{BAD} = \widehat{CED}$, so we have $\widehat{CAD} = \widehat{CED}$. It follows that triangle AEC is isosceles (**23**, converse theorem), and $AC = EC$. Since $AB = EC$ and $AC = EC$, we have $AB = AC$.

Notes. The level of difficulty of 3° is significantly greater than that of 1° or 2°. In 3°, we cannot prove the triangles congruent that are formed by the line in question (because we have the wrong corresponding parts). This may be the first time students encounter the case SSA, which does not allow them to conclude that two triangles are congruent. The full story of this case is investigated in a course in trigonometry (the "ambiguous case").

Exercise 6. *On side OX of some angle, we take two lengths OA, OB, and on side OX' we take two lengths OA', OB', equal to the first two lengths, respectively. We draw AB', BA', which cross each other. Show that point I, where these two segments intersect, lies on the bisector of the given angle.*

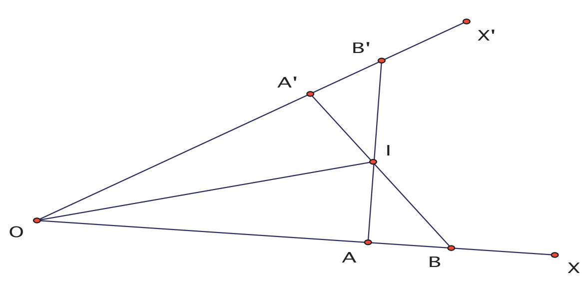

FIGURE t6

Solution. The statement will be proved by recognizing in turn three pairs of congruent triangles: OAB' and $OA'B$; then IAB and $IA'B'$; and finally OIA and OIA' (Figure t6).

First, triangles OAB' and $OA'B$ are congruent since they have a common angle at O and equal pairs of sides OA, OA' and OB', OB. It follows that $\widehat{OB'A} = \widehat{OBA'}$. For the same reason, angles $\widehat{OAB'}$, $\widehat{OA'B}$ are equal, so their supplements are equal, and $\widehat{B'A'B} = \widehat{BAB'}$. Also, sides $OB = OB'$ and $OA = OA'$. Subtracting, we find that $A'B' = AB$.

Therefore, triangles IAB and $IA'B'$ are congruent (ASA, **24**), so $IA = IA'$.

Finally, triangles OIA and OIA' are congruent (SSS), so $\widehat{AOI} = \widehat{A'OI}$. Thus, point I is on the bisector of angle $\widehat{AOB}$.

Exercise 7. *If two sides of a triangle are unequal, then the median between these two sides makes the greater angle with the smaller side. (Imitate the construction in* **25***.)*

Solution. Suppose the given triangle is ABC, with $BC > AB$, and let BD be a median. We extend BD to E so that $BD = DE$ (Figure t7). Triangles DAE, DCB are congruent (SAS, **25**), so $AE = BC$ and $\widehat{DEA} = \widehat{DBC}$.

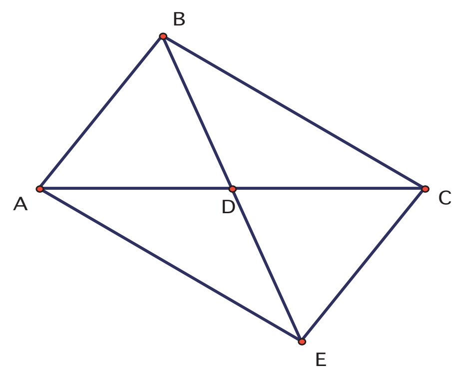

Figure t7

Since $BC > AB$ by assumption, it now follows that $AE > AB$ in triangle ABE. Then, from the theorem of **25**, $\widehat{ABE} > \widehat{AEB}$; that is, $\widehat{ABD} > \widehat{DBC}$.

Notes. The technique of extending a median past the midpoint by its own length (here, extending BD to E so that $BD = DE$) is one that is often useful in working with these line segments.

Exercise 8. *If a point in the plane of a triangle is joined to the three vertices of the triangle, then the sum of these segments is greater than the semiperimeter of the triangle; if the point is inside the triangle, the sum is less than the whole perimeter.*

Solution. For any point M in the plane of triangle ABC, we have $MB+MC \geq BC$, $MC + MA \geq CA$, $MA + MB \geq AB$. Equality in all three statements is not possible, since this would mean that point M is on all three lines BC, CA, AB at once (by Corollary II in **26**). Adding these three inequalities, we find that $MA + MB + MC > \frac{1}{2}(BC + CA + AB)$.

The perimeter of a convex figure is less than the perimeter of a figure surrounding it (see **27**). It follows that for any point M lying inside the triangle, we have $MB+MC < AB+AC$, $MC+MA < BC+BA$, $MA+MB < AC+BC$. Adding these inequalities and dividing by 2, we have $MA+MB+MC < BC+CA+AB$.

Notes. This is the beginning of a series of exercises which make excellent environments for exploration with dynamic geometry software.

An open-ended version of this problem is particularly useful. For example, students can be asked to construct any triangle, find the perimeter, then look for a lower bound for the sum $MA + MB + MC$. (It may help to ask students to calculate the perimeter of the triangle.)

The solutions given above are more sophisticated than those most students think of. Our solutions require familiarity with some of the more complicated inequality results previously proven. In particular, students often find the proof of the second proposition in the exercise challenging. One way to help them is to extend one of the segments, say MA, to side BC, and work with the resulting triangles. In effect, the students will be proving the result of **27**.

It is beyond the scope of Hadamard's text, but the first statement here is clearly true even if point M is chosen outside the plane of triangle ABC.

Students using dynamic geometry software can experiment with the sum of the distances of a point to a fixed triangle. They may naturally ask, for instance, where the minimal sum occurs. This is not a simple problem. See Exercises 8b and 105, and also **363**.

Exercise 8b. *If a point in the plane of a polygon is joined to the vertices of the polygon, then the sum of these segments is greater than half the perimeter of the polygon.*

Solution. If $ABC\dots KL$ is any polygon and M is the chosen point, then we have $MA+MB \geq AB$, $MB+MC \geq BC$, $MC+MD \geq CD$, $\dots$, $ML+MA \geq LA$, and it is impossible for equality to hold in all the statements simultaneously. Adding and dividing by 2, we get $MA + MB + \cdots + ML > \frac{1}{2}(AB + BC + \cdots + LA)$.

Notes. This is another good problem for exploration with geometry software. However, many students will get the solution by analogy with Exercise 8, without doing exploratory work.

The analogous result for a point inside a general polygon is also interesting. For a point inside a general polygon with n sides, the sum of the distances to the vertices is less than $\frac{n-1}{2}$ times the perimeter. The proof is analogous to the corresponding statement in Exercise 8. It is easy, but important, for students to see that this result reduces to the previous statement for the case $n = 3$.

Exercise 9. *The sum of the diagonals of a [convex] quadrilateral is between half the perimeter and the whole perimeter.*

Solution. If E is the point of intersection of diagonals AC and BD of quadrilateral $ABCD$ (Figure t9a), then $AC < AB + BC$, $AC < AD + DC$, $BD < BC + CD$, $BD < BA + AD$. Adding and dividing by 2, we find that $AC + BD < AB + BC + CD + DA$.

If quadrilateral $ABCD$ is convex, we can also get a lower bound for the sum of the diagonals. We have, for a convex quadrilateral, $AC = AE + EC$, $BD =$

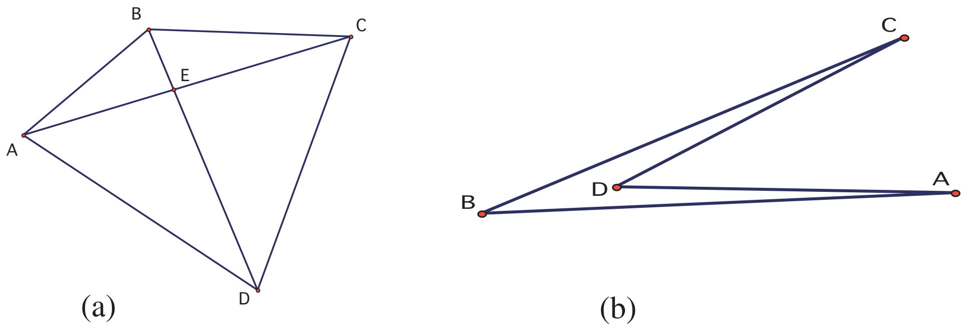

FIGURE t9

$BE+ED$ and $AE+EB > AB$, $BE+EC > BC$, $CE+ED > CD$, $DE+EA > AD$. Adding these last inequalities and dividing by 2, we have $AE+BE+CE+DE > \frac{1}{2}(AB+BC+CD+AD)$ or $AC+BD > \frac{1}{2}(AB+BC+CD+AD)$.

Notes. The property that $AC + BD < AB + BC + CD + DA$ holds true even for a concave quadrilateral. The second property, that $AC + BD > \frac{1}{2}(AB + BC + CD + DA)$ is not always true for a concave quadrilateral. For example, in Figure t9b, $AC + BD$ is clearly less than half the perimeter of the quadrilateral.

Exercise 10. *The intersection point of the diagonals of a [convex] quadrilateral is the point in the plane such that the sum of its distances to the four vertices is as small as possible.*

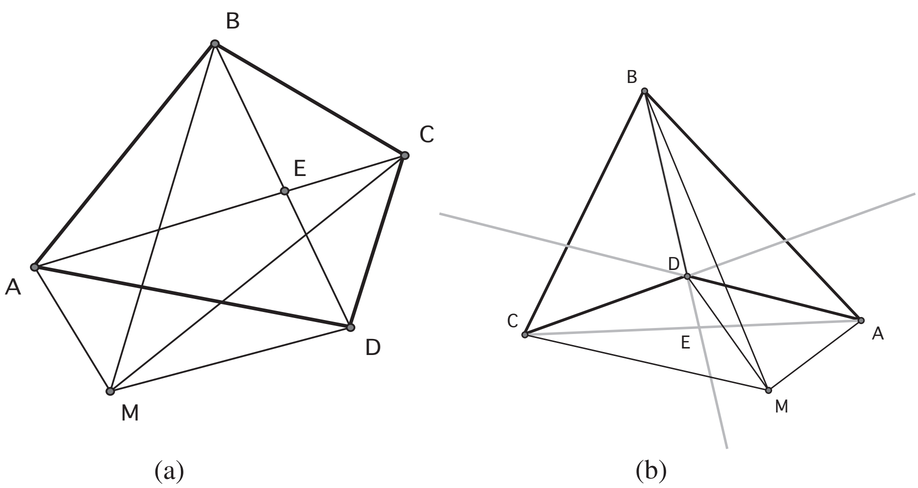

FIGURE t10

Solution. Suppose E is the intersection point of the diagonals of convex quadrilateral $ABCD$, and let M be any point on the plane of the quadrilateral other than E (Figure t10a). We have $MA+MC \geq AC = AE+EC$, $MB+MD \geq BD =$

$BE+ED$, and, since point M is distinct from point E, equality cannot hold in both statements at once. Adding these inequalities, we have $MA+MB+MC+MD > EA+EB+EC+ED$.

Notes. The point of intersection E of the diagonals of a concave quadrilateral does not have the property shown above. To see this, let D be that vertex of a concave quadrilateral (Figure t10b) which lies inside the triangle ABC formed by the other three vertices. If point M lies inside (or on the sides of) the angle vertical to angle BDC (formed by extending segments BD and CD past D), then **(27)** $MB+MC > DB+DC$. Also, $MD+MA \geq AD$. It follows that

$$MA+MB+MC+MD > DA+DB+DC. \tag{1}$$

The same inequality holds if point M is inside (or on) the angle vertical to angle ADB or inside the angle vertical to angle ADC. Thus, inequality (1) holds for any point M other than D (and in particular for the point E of intersection of the diagonals in the figure). A simple dynamic sketch will make this phenomenon clear to the student.

In this case, point D, and not E, is the point on the plane for which the sum of the distances to the vertices is smallest.

Exercise 11. *A median of a triangle is smaller than half the sum of the sides surrounding it and greater than the difference between this sum and half of the third side.*

Solution. In Figure t11, we see triangle ABC with median BD extended its own length to E. Triangles ADE, CDB are congruent by SAS, so $AE = BC$. In triangle ABE, $2BD < AB+AE$ or $BD < \frac{1}{2}(AB+BC)$.

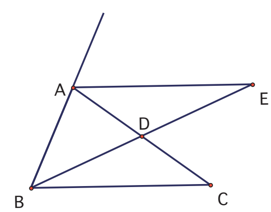

FIGURE t11

In triangles ABD and BCD we have $BD > AB - AD$ and $BD > BC - DC$. Adding and dividing by 2, we have $BD > \frac{1}{2}(AB+BC-AC)$.

Notes. Setting up an exploration with software can be tricky for this problem, but it provides significant insight. For instance, the students can construct a *hinged* triangle, with two fixed sides. They can do this by first constructing a segment, which will be one side of the triangle. Then they can construct an arbitrary circle centered at one of the endpoints of the first segment. Choosing a point on this circle and connecting this point to its center provides a second side of the triangle. Then they can construct the third side and the median to that side. Moving the point along the circle will give them the desired hinged triangle.

Even without measuring the median or the perimeter, it is now easy to see that the maximum and minimum length of the median of the triangle is determined by the situation when the two hinged sides are nearly collinear. When the center is "almost" outside the two sides, the median is shortest, and it is not hard to see that it is almost equal to half the difference of these sides. (Students who don't see this from the diagram can be given the hint to use software to compare the length of the median to half the difference of the sides.)

When the circle's center is "almost" on the segment determined by the two sides, the median is longest. In this case, it is difficult to see from the diagram how this minimal value is related to the lengths of the sides of the triangle. Again, students can be given the hint to compute half the sum of two sides minus half the third side. They may be surprised to see that this last quantity is zero when the three vertices of the "triangle" are collinear.

Exercise 11 is typical of problems about geometric inequalities, in that the exploration (using software) is essentially analytic in nature, while the solution is algebraic. That is, in moving elements of a drawing around, we are exploiting the continuous variation of the quantities in question, which is an analytic property of the functions concerned. If we were teaching calculus, we would harness that tool to find maxima and minima.

But in writing a formal solution, in the tradition of Euclidean geometry, we are performing algebraic operations and writing inequalities in algebraic form. The analytic nature of the problem disappears. This is typical of elementary treatments of inequalities: their analytic nature is disguised by the algebraic tools we use with them.

See Exercise 12 for an application of this result.

Exercise 12. *The sum of the medians of a triangle is greater than its semi-perimeter and less than its whole perimeter.*

Solution. We use the result of Exercise 11 (this can be given to students as a hint). If AA', BB', and CC' are medians of triangle ABC, then this result gives us $\frac{1}{2}(AB + AC - BC) < AA' < \frac{1}{2}(AB + AC)$, $\frac{1}{2}(AB + BC - AC) < BB' < \frac{1}{2}(AB + BC)$, and $\frac{1}{2}(AC + BC - AB) < CC' < \frac{1}{2}(AC + BC)$. Adding, we find that $\frac{1}{2}(AB + AC + BC) < AA' + BB' + CC' < AB + AC + BC$.

Notes. Using the theorem of **56**, we can prove that in fact $\frac{3}{4}(AB+BC+AC) < AA' + BB' + CC'$, which gives a better lower bound to the sum of the medians. Indeed, if G is the intersection point of the medians, then $BC < GB + GC$ or $BC < \frac{2}{3}(BB' + CC')$. Similarly, $AC < \frac{2}{3}(AA' + CC')$ and $AB < \frac{2}{3}(AA' + BB')$. Adding, we quickly find that $\frac{3}{4}(AB + BC + AC) < AA' + BB' + CC'$.

Exercise 13. *On a given line, find a point such that the sum of its distances to two given points is as small as possible. Distinguish two cases, according as whether the points are on the same side of the line or not. The second case can be reduced to the first (by reflecting part of the figure in the given line).*

Solution. If the given points A and B lie on opposite sides of the given line XY, then the required point is the point M of intersection of XY and AB. Indeed, if M' is a different point on XY, then (in triangle $AM'B$) $MA + MB = AB < M'A + M'B$.

If the given points A and B are on the same side of XY, then take point B' symmetric to B with respect to XY. For any point M' on XY, we have

$M'A + M'B = M'A + M'B'$, so the required point is the intersection M of XY and AB'.

Notes. This and the next exercise are classics. For students who have not seen these, the solutions are not at all obvious, yet the terms of the problem are clear. This makes for good introductory problem solving—even before students have the tools for the rather sophisticated solutions we give.

These problem situations also lead naturally to a discussion of the nature of proof. The usual guesses (the midpoint of the projections or a perpendicular from one of the points to the line) are in general wrong, and considerations of special cases do not usually give students the required insight. Instead, students here must use logic to deduce the correct solution. If they do so (which is rare), they can usually give a proof very quickly.

With geometric software, they can often get to the solution more quickly but without the insight necessary for proof. That is, they can approximate the minimal point of intersection, but not characterize it. Describing this point requires an "extra" step. One effective way to combine these two approaches in the classroom is to have a discussion without software, then ask students to make conjectures and check them with the software.

A tricky pedagogical situation can arise if students notice (from their software experiments) that the minimal path makes equal angles with the given line. The problem is that it is not easy to use this insight to achieve a proof. In our solution, the equality of the angles follows from the proof that this path, which is characterized slightly differently, is minimal.

This exercise generalizes in several ways. One way is given in Exercise 40 below. Students can also think about what happens in space if the line is replaced by a plane (reflection in a plane takes the place of reflection in the line) or if the two given points are not coplanar with the given line (the situation can be reduced to that of two dimensions).

This problem is also related to the reflection property of an ellipse: If we draw a tangent to the ellipse, the lines joining the foci to the point of tangency will make equal angles with the tangent. If the ellipse is a circle, this reduces to the fact that a tangent is perpendicular to a radius drawn to the point of tangency.

Exercise 14. *(Billiard problem.) Given a line XY and two points A, B on the same side of the line, find a point M on this line such that $\widehat{AMX} = \widehat{BMY}$. We obtain the same point as in the preceding exercise.*

Solution. If B' is the point symmetric to B with respect to XY, then $\widehat{BMY} = \widehat{B'MY}$ (Figure t11). Since $\widehat{AMX} = \widehat{B'MY}$, it follows that $\widehat{AMX} + \widehat{XMB'} = \widehat{XMB'} + \widehat{B'MY}$. This shows that points A, M, B' are collinear or that the required point is the intersection of AB' and XY.

Exercise 15. *On a given line, find a point with the property that the difference of its distances to two given points is as large as possible.*

Distinguish two cases, according as whether the points are on the same side of the line or not.

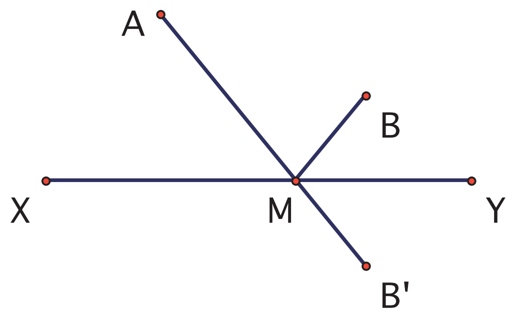

Figure t14

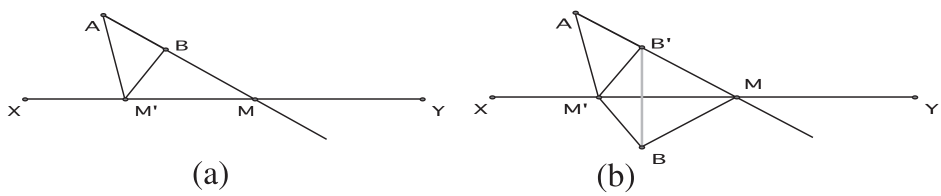

Figure t15

Solution. If the given points A and B lie on the same side of the given line XY (Figure t15a), then the required point is the intersection M of XY and AB. Indeed, if M' is any point on line XY other than M, we have $|MA - MB| = AB > |M'A - M'B|$.

If A and B lie on opposite sides of XY (Figure t15b), then let B' be the point symmetric to B with respect to XY. Then for any point M' on line XY we have $M'A - M'B = M'A - M'B'$. Therefore, the required point M is the intersection of XY and AB'.

Exercise 16. *If the legs of a first right triangle are respectively smaller than the legs of a second, then the hypotenuse of the first is smaller than the hypotenuse of the second.*

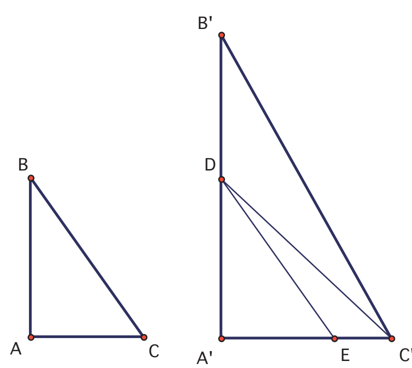

Figure t16

Solution. Suppose the two triangles are ABC and $A'B'C'$ (Figure t16) with right angles at A and A', and suppose that $AB < A'B'$ and $AC < A'C'$. We can construct $A'D = AB$ and $A'E = AC$, with D, E on sides $A'B'$ and $A'C'$, respectively. Then triangles ABC and $A'DE$ are congruent (SAS), and $BC = DE$. Since $A'E < A'C'$, it follows (from **29**, 3°) that $DE < DC'$. Also from **29**, 3°, we see that $DC' < B'C'$, so that $DE < B'C'$, or $BC < B'C'$.

Exercise 17. *If the angles* $\widehat{B}$ *and* $\widehat{C}$ *of a triangle* ABC *are acute and the sides* AB, AC *unequal, then the lines starting from* A *are encountered in the following order: larger side, median, bisector, altitude, smaller side.*

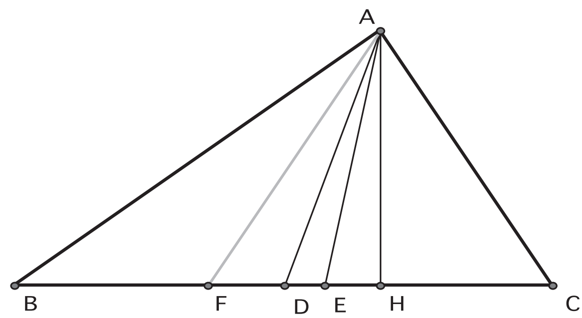

FIGURE t17

Solution. In triangle ABC, let AD be a median, let AE be an angle bisector, and let AH be an altitude. We assume that $AB > AC$ (Figure t17). By the result of Exercise 7, we have $\widehat{BAD} < \widehat{DAC}$, so median AB lies between side AB and angle bisector AE.

Now we locate point F on BC such that $HF = HC$. Then (again by **29, 30**) $AF = AC$, so $AB > AF$, and therefore $BH > FH$; that is, F is closer to H than B is. Thus, $\widehat{CAH} = \widehat{FAH} < \widehat{BAH}$. Thus AH makes a smaller angle with AC than it does with AB, so AH lies between angle bisector AE and side AC. This proves the assertion of the problem.

Exercise 18. *The median of a nonisosceles triangle is greater than the bisector from the same vertex, bounded by the third side.*

Solution. Since $HE < HD$ (see the solution to Exercise 17, and Figure t17), it is also true that $AE < AD$.

Exercise 19. *Show that a triangle is isosceles if it has two equal altitudes.*

Solution. In triangle ABC, if altitudes BD and CE are equal (Figure t19), then right triangles BCD and BCE are equal by hypotenuse-leg (**34**). Therefore, $\widehat{BCD} = \widehat{CBE}$, and triangle ABC is isosceles.

Notes. The analogous statement for the medians of a triangle is given in Exercise 39 and for the angle bisectors in Exercises 361 and 361a.

Exercise 20. *More generally, in any triangle, the greater side corresponds to the smaller altitude.*

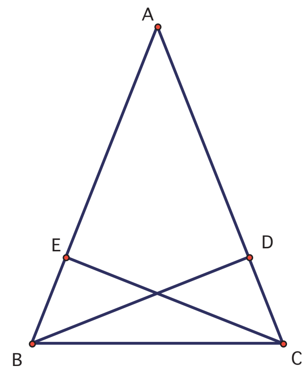

Figure t19

Solution. In triangle ABC, suppose $AB > AC$. Then $\widehat{ACB} > \widehat{ABC}$. If BD and CE are altitudes of the triangle, then we apply the theorem of **35** to find that $BD > CE$.

Exercise 21. *In a triangle ABC, consider a parallel to BC passing through the intersection point of the bisectors of $\widehat{B}$ and $\widehat{C}$. This parallel intersects AB in M and BC in N. Show that $MN = BM + CN$. What happens to this statement if the parallel is drawn through the intersection point of the bisectors of the exterior angles at B and C? Or through the intersection of the bisector of $\widehat{B}$ with the bisector of the exterior angle at C?*

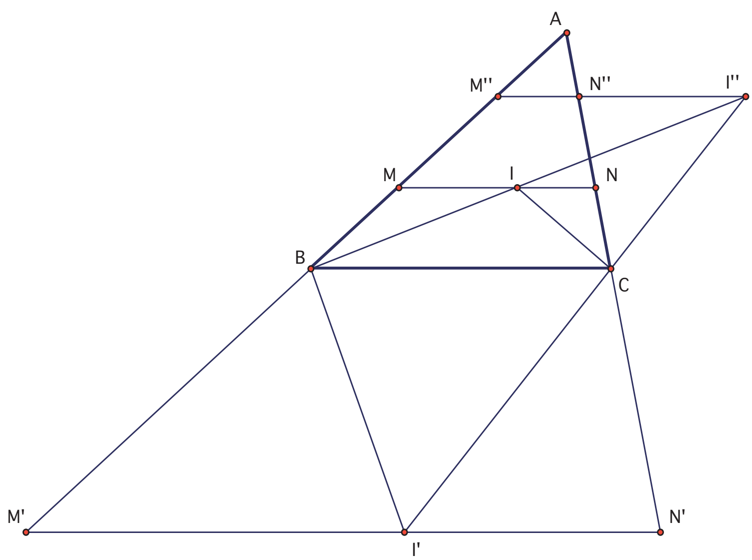

Figure t21

Solution. Suppose I is the intersection point of the bisectors of angles $\widehat{B}$ and $\widehat{C}$ (Figure t21). We have $\widehat{NCI} = \widehat{ICB} = \widehat{NIC}$ (they are alternate interior angles of parallel lines), so triangle NIC is isosceles, and $IN = CN$. Similarly, $IM = BM$, so that $MN = BM + CN$.

Next let I' be the point of intersection of the exterior angles at B and C, and let $M'I'N'$ be the line parallel to BC through I'. Then triangles $M'BI'$ and $N'CI'$ are both isosceles, and $M'N' = BM' + CN'$.

Finally, suppose I'' is the intersection point of the bisectors of the interior angle at B and the exterior angle at C, and let $M''I''N''$ be the line parallel to BC through I''. Then triangles $M''BI''$ and $N''CI''$ are isosceles, so $M''N'' = M''I'' - N''I'' = BM'' - CN''$.

More accurately, $M''N'' = |M''I'' - N''I''| = |BM'' - CN''|$ because it could happen that M'', N'' are on the extensions of AB, AC past A.

Notes. These seemingly easy problems are in fact often difficult for students. The difficulty lies in the orientation of the isosceles triangles. Students naturally draw the original triangle with one side parallel to the bottom of the page they are working on. The lines of symmetry of the isosceles triangles in question are then oblique to this direction. This obscures the symmetry of the two isosceles triangles so they are not recognized. Sketching in the angle bisectors adds another difficulty: students are typically not good at estimating the "middle" of an angle. Constructing the figure (with hand construction tools or with software) may make the second point easier. Nonetheless, this and the following problems tend to be remarkably difficult for students to solve.

One way to present this series of problems to students might be to give them only the first one (with the interior angle bisectors). Then, even if they must be shown the solution, they can be asked to state and prove a similar theorem for two exterior angle bisectors or for one interior and one exterior angle bisector. Working with geometry software can help them to formulate the theorems, after which the proofs are analogous to the one they were shown.

Another way to extend this theorem is to form various converses. Basically, this theorem concerns three statements:

(i) MN is parallel to BC;

(ii) MN contains the point I at which two angle bisectors intersect;

(iii) $MN = BM + NC$.

Part 1 of the given problem asks students to prove that (i) and (ii) imply (iii). Students can investigate whether (i) and (iii) imply (ii), or whether (ii) and (iii) imply (i).

They can also note that one of these conditions alone is not enough to imply the other two.

Exercise 22. *Prove the theorem of* **44b** *by decomposing the polygon into triangles using segments starting from an interior point of the polygon.*

Solution. If we are given polygon $ABCDE$ (Figure t22), we can choose a point O in its interior, and connect it to each vertex. We will then decompose the polygon into as many triangles as it has sides. The sum of the angles of these triangles will be $n(180°)$ (where n is the number of sides in the polygon). To get the sum of the interior angles of the polygon, we must subtract the sum of the angles around point O, which is $2(180°)$. The result follows.

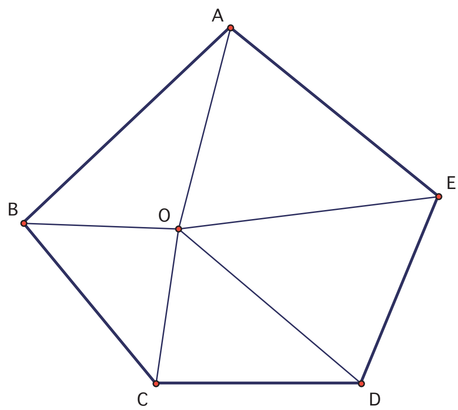

FIGURE t22

For this solution to be correct, the polygon must be *starlike*: there must be a point inside it from which every vertex is visible. We have chosen such a point for point O. Any convex polygon, which is the subject of **44b**, is starlike. The theorem is in fact true for any polygon, but the proof involves a bit more than the arguments given here.

Exercise 23. *In triangle ABC we draw lines AD, AE from point A to side BC, such that the first makes an angle equal to $\widehat{C}$ with AB, while the second makes an angle equal to $\widehat{B}$ with AC. Show that triangle ADE is isosceles.*

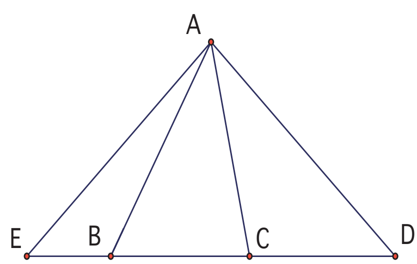

FIGURE t23a

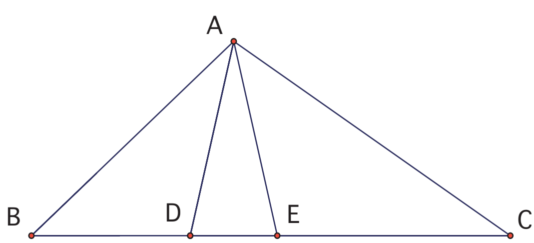

FIGURE t23b

Solution. If the angle at A is acute, then the sum $\widehat{B}+\widehat{C}$ will be greater than angle $\widehat{A}$ (Figure t23a). From triangle ABD, we have $\widehat{ADE} = 180° - \widehat{B} - \widehat{C} = \widehat{A}$.

Similarly, from triangle ACE we have $\widehat{AED} = 180° - \widehat{B} - \widehat{C} = \widehat{A}$, so $\widehat{ADE} = \widehat{AED}$.

If the angle at A is obtuse, then $\widehat{B}+\widehat{C} < \widehat{A}$ (Figure t23b). From triangles ABD, ACE we have (**44**, Corollary I) $\widehat{ADE} = \widehat{B}+\widehat{C}$, $\widehat{AED} = \widehat{B}+\widehat{C}$. Therefore $\widehat{ADE} = \widehat{AED}$.

Finally, if the angle at A is a right angle, then points D and E coincide since $\widehat{A} = \widehat{B}+\widehat{C}$.

Notes. Because this diagram is so difficult to draw freehand, this is an excellent exercise in the use of geometric software. Students will find that if they draw the diagram properly, Figures t23a and t23b can be obtained from the same software sketch.

Exercise 24. *In any triangle ABC we have:*

1°. *The bisector of $\widehat{A}$ and the altitude from A make an angle equal to half the difference between $\widehat{B}$ and $\widehat{C}$.*

2°. *The bisectors of $\widehat{B}$ and $\widehat{C}$ form an angle equal to $\frac{1}{2}\widehat{A}$ + one right angle.*

3°. *The bisectors of the exterior angles of $\widehat{B}$ and $\widehat{C}$ form an angle equal to one right angle - $\frac{1}{2}\widehat{A}$.*

Solutions. 1°. In triangle ABC, let AE be the angle bisector and AH the altitude, both from vertex A. Suppose $AB > AC$, so that $\widehat{B}$ is acute (Figure t17). Then $\widehat{EAH} = \widehat{BAH} - \widehat{BAE} = 90° - \widehat{B} - \frac{1}{2}(180° - \widehat{B} - \widehat{C}) = \frac{1}{2}(\widehat{C} - \widehat{B})$.

2°. If I is the intersection point of the bisectors of $\widehat{B}$ and $\widehat{C}$ (Figure t21), then $\widehat{BIC} = 180° - (\frac{1}{2}\widehat{B} + \frac{1}{2}\widehat{C}) = 180° - \frac{1}{2}(180° - \widehat{A}) = 90° + \frac{1}{2}\widehat{A}$.

3°. If I' is the intersection point of the bisectors of the exterior angles at B and C, then (Figure t21) $\widehat{BI'C} = 180° - \widehat{M'I'B} - \widehat{N'I'C} = 180° - \frac{1}{2}(180° - \widehat{B}) - \frac{1}{2}(180° - \widehat{C}) = \frac{1}{2}(180° - \widehat{A}) = 90° - \frac{1}{2}\widehat{A}$.

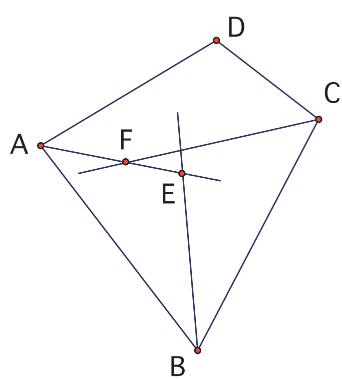

FIGURE t25

Exercise 25. *In a convex quadrilateral:*

1°. *The bisectors of two consecutive angles form an angle equal to one half the sum of the other two angles.*

2°. *The bisectors of two opposite angles form an angle supplementary to half the difference of the other two angles.*

Solutions. 1°. If AF and BE are the bisectors of angles $\widehat{A}$ and $\widehat{B}$ (Figure t25), then triangle ABE gives us $\widehat{AEB} = 180° - \frac{1}{2}\widehat{A} - \frac{1}{2}\widehat{B} = \frac{1}{2}(\widehat{A}+\widehat{B}+\widehat{C}+\widehat{D}) - \frac{1}{2}\widehat{A} - \frac{1}{2}\widehat{B} = \frac{1}{2}(\widehat{C}+\widehat{D})$.

2°. Let the bisectors of angles $\widehat{A}$, $\widehat{C}$ intersect at F. Of the two quadrilaterals $ABCF$ and $ADCF$, one must be convex. Let us assume that this is the one labeled $ADCF$, so that the diagram looks like Figure t25. Then quadrilateral $ADCF$ gives us $\widehat{AFC} = 360° - \widehat{D} - \frac{1}{2}\widehat{A} - \frac{1}{2}\widehat{C} = 360° - \widehat{D} - \frac{1}{2}(360° - \widehat{B} - \widehat{D}) = 180° - \frac{1}{2}(\widehat{D} - \widehat{B})$.

Notes. In 1°, the bisectors AF, BE will always intersect. But in 2° the bisectors of angles $\widehat{A}$, $\widehat{C}$ might be parallel. Students can investigate this case for themselves.

Exercise 26. *The angle bisectors of a parallelogram form a rectangle. The bisectors of the exterior angles also form a rectangle.*

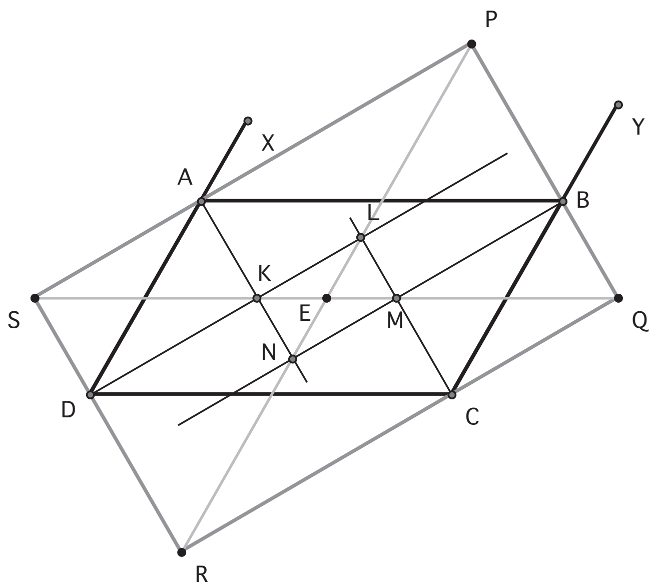

FIGURE t26

Solution. If AN and BN are the bisectors of angles $\widehat{A}$ and $\widehat{B}$ in parallelogram $ABCD$ (Figure t26), then $\widehat{ANB} = 180° - \frac{1}{2}\widehat{A} - \frac{1}{2}\widehat{B} = 180° - 90° = 90°$, since the sum of two consecutive angles of a parallelogram is 180°. If AP and BP are the bisectors of the exterior angles at A and B, then $\widehat{APB} = 180° - \frac{1}{2}\widehat{XAB} - \frac{1}{2}\widehat{YBA} = 90°$. Similarly, we can show that the other angles of quadrilaterals $KLMN$ and $PQRS$ are right angles.

Notes. Students making a dynamic sketch for this problem can explore many other properties. For example:

- the sides of the two rectangles are parallel in pairs;
- the centers of the rectangles are both at the center of the original parallelogram;
- the vertices of the inner rectangle lie on the diagonals of the outer rectangle.

Several other relationships may also become apparent as students use a dynamic sketch to explore the situation. They can then be asked to prove their assertions. See also Exercises 41 and 109.

Exercise 27. *Any line passing through the intersection of the diagonals of a parallelogram is divided by this point, and by two opposite sides, into two equal segments.*

For this reason, the point of intersection of the diagonals of a parallelogram is called the center *of this polygon.*

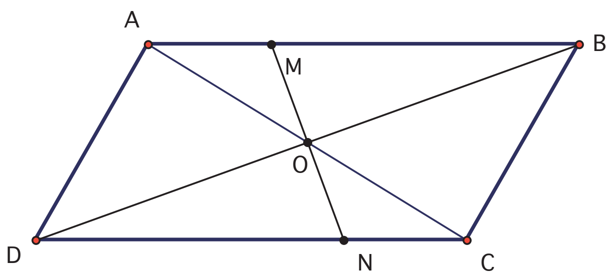

FIGURE t27

Solution. If O is the intersection point of the diagonals of parallelogram $ABCD$ (Figure t27) and MN is any line passing through O, then triangles AOM and CON are congruent (since $AO = OC$, $\widehat{MAO} = \widehat{NCO}$, and $\widehat{AOM} = \widehat{CON}$), so $OM = ON$.

Exercise 28. *Two parallelograms, one of which is* inscribed *in the other (that is, the vertices of the second are on the sides of the first) must have the same center.*

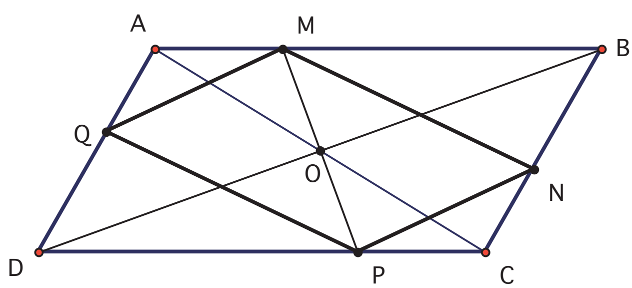

FIGURE t28

Solution. The proposition will be proved if we can show that the midpoint of a diagonal of the smaller parallelogram coincides with a midpoint of a diagonal of the larger.

To this end, let O be the midpoint of diagonal MP of parallelogram $MNPQ$, which is inscribed in parallelogram $ABCD$ (Figure t28). Triangles AMQ and CPN are congruent, since $MQ = PN$, $\widehat{AMQ} = \widehat{CPN}$, and $\widehat{AQM} = \widehat{CNP}$ (**43**). It follows that $AM = CP$. Also, triangles AOM and COP are congruent, since $AM = CP$, $OM = OP$, and $\widehat{AMO} = \widehat{CPO}$. It follows that $\widehat{AOM} = \widehat{COP}$ and $OA = OC$. Since $\widehat{AOP} = \widehat{MOC}$ (**12**), this means that AOC is a straight line (see Exercise 3). Therefore, O is the midpoint of diagonal AC of parallelogram $ABCD$.

Exercise 29. *An angle of a triangle is acute, right, or obtuse, according as whether its opposite side is less than, equal to, or greater than double the corresponding median.*

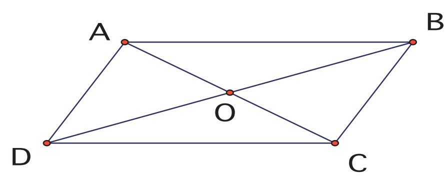

FIGURE t29

Solution. If BO is a median of triangle ABC and $BO = OD$ (Figure t29), then we can apply the theorem of **28** (concerning two triangles with two pairs of equal sides, but unequal included angles) to triangles ABC and ABD. Depending on whether $AC < 2OB$, $AC = 2OB$, or $AC > 2OB$, we have $AC < BD$, $AC = BD$, $AC > BD$, and therefore $\widehat{ABC} < \widehat{BAD}$, $\widehat{ABC} = \widehat{BAD}$, $\widehat{ABC} > \widehat{BAD}$. Since $\widehat{ABC} + \widehat{BAD} = 180°$, we have $\widehat{ABC} < 90°$, $\widehat{ABC} = 90°$, $\widehat{ABC} > 90°$, respectively.

Notes. Students may enjoy seeing this result phrased for a parallelogram: an angle of a parallelogram is acute, right, or obtuse depending on whether the diagonal not passing through the vertex of that angle is greater than, equal to, or less than the other diagonal of the figure. (The proof, and even the diagram, is virtually identical.)

Exercise 30. *If, in a right triangle, one of the acute angles is double the other, then one of the sides of the right angle is half the hypotenuse.*

Solution. In right triangle ABC, with hypotenuse BC (Figure t30), suppose $\widehat{B}$ is twice $\widehat{C}$. If we extend segment AB to D so that $AB = AD$, then all the angles of triangle BCD will be equal, which means that $BC = BD = 2BA$.

Notes. Both the original statement about a 30°-60°-90° triangle and this converse result are important in that they foreshadow a lot of trigonometry and metric geometry to come. Essentially, this is the statement that the value of $\sin 30°$ is $\frac{1}{2}$. It is important for students to see that the reason this happens is that a 30°-60°-90° triangle is half of an equilateral triangle.

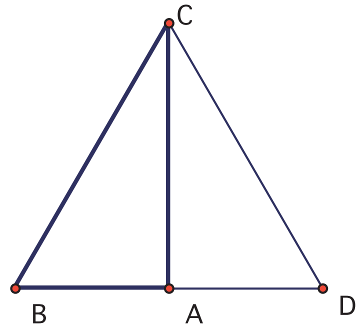

FIGURE t30

One way to get students to appreciate that this triangle is special is to explicitly perform the construction in the proof of the original statement but starting with, say, a 20°-70°-90° triangle. Students will find that they cannot draw the same conclusion: the triangle they get is isosceles, but not equilateral.

Exercise 31. *Find the locus of the points such that the sum or difference of its distances to two given lines is equal to a given length.*

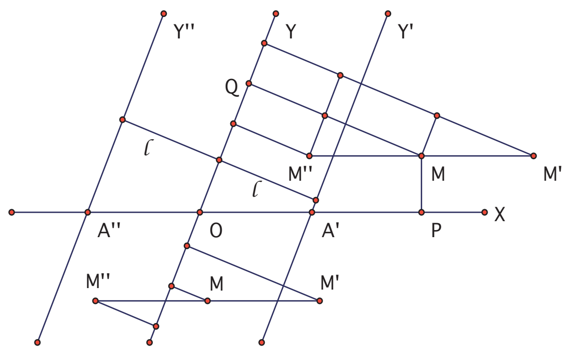

FIGURE t31a

Solution. Suppose the given lines are OX and OY (Figure t31a). Let A' and A'' be the intersection points of line OX with lines $A'Y'$ and $A''Y''$, parallel to OY and at a distance from it equal to the given length l (see Corollary II to **51**).

Let M be any point on the plane, and suppose MP and MQ are the distance from M to lines OX and OY, respectively. (In the diagram, P and Q are the feet of the perpendiculars from M to OX and OY.) We translate M by the direction and distance of OA' to get point M', and by the direction and distance of OA'' to get point M''. The distance from each of these two new points to OX remains the same: it is equal to MP.

The distance from each of these new points to OY are either $MQ + l$, $l - MQ$, or $MQ + l$, $MQ - l$, depending on whether or not point M lies between lines $A'Y'$ and $A''Y''$. (Both situations are show in Figure t31a).

Now suppose, for point M, one of the following conditions holds:

- $MP + MQ = l$; i.e., $MP = l - MQ$;
- $MP - MQ = l$; i.e., $MP = MQ + l$;
- $MQ - MP = l$; i.e., $MP = MQ - l$.

Then one of the points M', M'' must be equidistant from lines OX and OY. Following this logic backwards, we find that any point M satisfying one of the conditions of the problem can be obtained by a translation through OA' and OA'' of the set of points equidistant from the given lines.

The set of points equidistant from the two intersecting lines OX, OY (Figure t31b) is the pair of lines p, q bisecting the vertical angles formed, so the set of points satisfying at least one of the conditions of the problem consists of two pair of parallel lines p', q' and p'', q'' obtained from OX, OY by the translations considered.

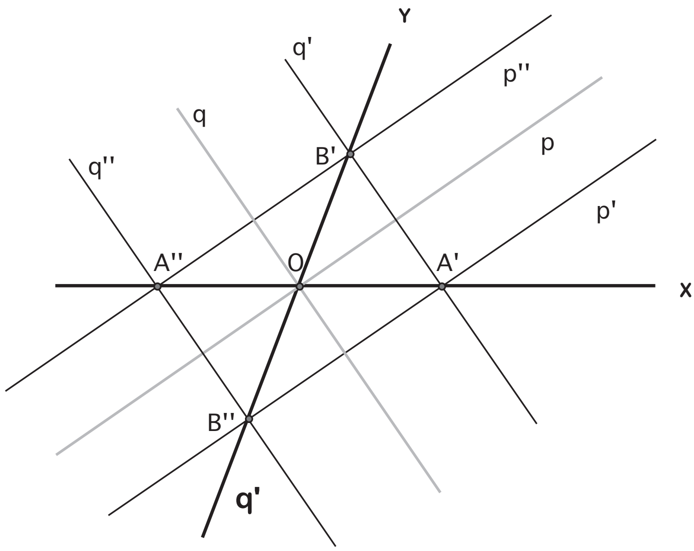

FIGURE t31b

We must now determine the set of points for which the segment l will be the sum of the distances to the given lines, and the set of points for which l will be the difference. We will answer this question for the points on line q': the answer will then be clear for other points. From Figure t31b it is not hard to see that the sum of the distances will be equal to l for those points on segment $A'B'$, and the difference of the distances will be equal to l for points on the extension of segment $A'B'$.

Thus, the set of points for which the sum of the distances to two given lines is equal to l is the four sides of rectangle $A'B'A''B''$, and the set of points for which the difference of the distances is equal to l is the union of the extensions of the four sides of this rectangle.

It remains to consider the case when the two lines are parallel. Suppose the given parallel lines are XX' and YY' (Figure t31c), and let ZZ' be a line parallel to and equidistant from both. Let MP, MQ, and MN be the distances from some point M to XX', YY', and ZZ', respectively.

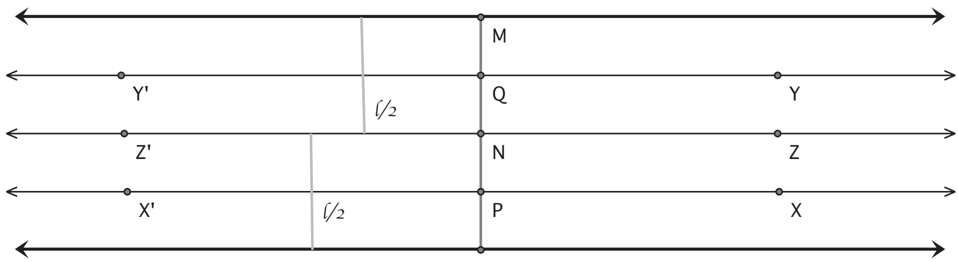

Figure t31c

If the distance between the given lines is less than l, then the set of points M for which $MP + MQ = l$ is a pair of lines at a distance of $\frac{1}{2}l$ from ZZ', because (from Exercise 1) we have $MP + MQ = 2MN$ (Figure t31c). In this case, there are clearly no points such that $MP - MQ = \pm l$.

If the distance between the given lines is greater than l, then clearly there are no points M such that $MP + MQ = l$. In this case, the set of points for which $MP-MQ = \pm l$ consists of two lines at a distance $\frac{1}{2}l$ from ZZ', since (from Exercise 1) we have $MP - MQ = \pm 2MN$ (Figure t31d).

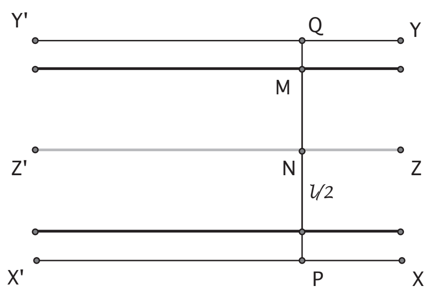

Figure t31d

Finally, if the distance between the given points is equal to l, then for any point M lying between the two given parallel lines or on either of them, we have $MP + MQ = l$. For any point lying outside the two lines or on one of them, we have $PM - MQ = \pm l$.

Exercise 32. *Given two parallel lines, and two points A, B outside these two parallels, and on different sides, what is the shortest broken line joining the two points, so that the portion contained between the two parallels has a given direction?*

Solution. Let the given lines be p and q (Figure t32), the given points A and B, and the direction given by line XY, where X and Y lie on p and q, respectively. We translate point B through XY (from X to Y), to get B'. Clearly, the segment AB' is the shortest distance from A to B'. Let N be the intersection of q with AB'. We claim now that if point M is on p so that MN is parallel to XY, then $ANMB$ is the required shortest path.

Indeed, take any other path $AN'M'B$ (in which $M'N'$ is parallel to XY), and translate BM' by XY. Point B again falls on B', while M' falls on N'. The length

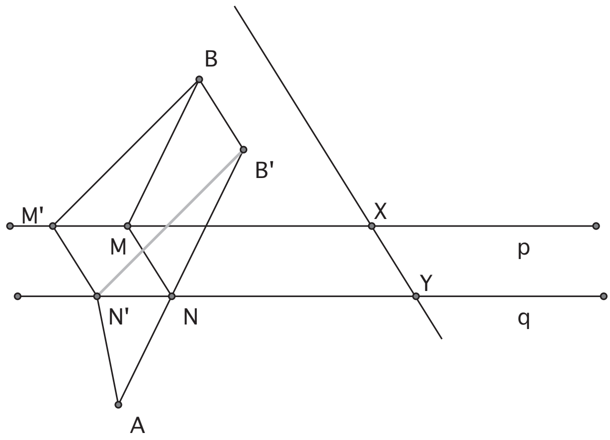

FIGURE t32

of the path $AN'M'B$ is equal to $AN' + N'B'$ plus the length of XY, while the length of path $ANMB$ is equal to $AN + NB' = AB'$ plus the length of XY. But the triangle inequality tells us that $AN' + N'B' > AB'$, so the path $AMNB$ is shorter.

Notes. As further exercise, students may be asked to show that one obtains just the same solution if we translate A (by YX) rather than B.

Students may also be asked to prove the analogous statement for two pairs of parallel lines and two given directions. They will have to translate one of the two given points twice. This problem generalizes Exercise 32 in just the same way that Exercise 40 generalizes Exercise 14.

Exercise 33. *Join a given point to the intersection of two lines, which intersect outside the limits of the diagram (see* **53***).*

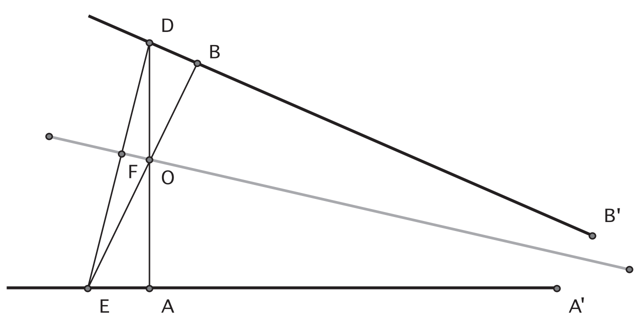

FIGURE t33

Solution. Following the hint to use **53**, we use the fact that the three altitudes of a triangle are concurrent.

Suppose we want to connect point O to the intersection of the lines labeled AA' and BB' in Figure t33. We draw lines AD and BE through O, perpendicular to AA' and BB', respectively (as shown in the diagram). Line OF, perpendicular to DE, will pass through the intersection point of AA' and BB' since OF is the third altitude of the triangle formed by EA', DB', and DE.

This construction works only if we assume that points A and B, as well as segment DE, lie within the limits of the diagram.

Students can draw for themselves the situation in which O lies outside the angle formed by lines AA', BB'. The construction remains valid in this case.

Exercise 34. *In a trapezoid, the midpoints of the nonparallel sides and the midpoints of the two diagonals are on the same line, parallel to the bases. The distance between the midpoints of the nonparallel sides equals half the sum of the bases; the distance between the midpoints of the diagonals is equal to half their difference.*

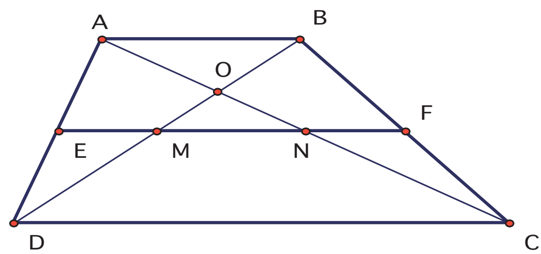

FIGURE t34

Solution. Suppose E and F are the midpoints of the nonparallel sides AD and BC of trapezoid $ABCD$, and suppose M and N are the midpoints of diagonals BD and AC (Figure t34). (We assume that the trapezoid is labeled so that $AB < CD$.) Line EM joins two midpoints of triangle ABD, and so is parallel to AB (**55**). Line EN joins two midpoints of triangle ACD, so is parallel to CD, and therefore to AB. It follows that lines EM and EN coincide. Points E, M, and N are thus collinear. Similarly, we can show that points F, M, and N are collinear. The line they lie on coincides with EMN, since they have points M and N in common. Thus, the four points E, F, M, and N are all collinear.

From triangles ABD, ABC, BCD we have

$$EM = \frac{1}{2}AB, \quad FN = \frac{1}{2}AB, \quad MF = \frac{1}{2}CD.$$

It follows that

$$EF = EM + MF = \frac{1}{2}(AB + CD),$$
$$MN = MF - FN = \frac{1}{2}(CD - AB).$$

Exercise 35. *If, from two points A, B and the midpoint C of AB, we drop perpendiculars onto an arbitrary line, the perpendicular from C is equal to half the sum of the other two perpendiculars or to half their difference, according as whether these two perpendiculars have the same or the opposite sense.*

Lemma 1. *A line through the midpoint of one leg of a trapezoid and parallel to the bases, bisects the other leg of the figure.*

Proof. Indeed, we know that the line joining the midpoints of the legs is parallel to the bases, and there is only one line parallel to the bases passing through a midpoint of one of the legs. Therefore, the line given in the statement of the lemma must be the unique parallel to the bases through the given midpoint.

In just the same way, we can prove a second lemma.

Lemma 2. *A line through the midpoint of one diagonal of a trapezoid, parallel to the bases, bisects the other diagonal of the figure.*

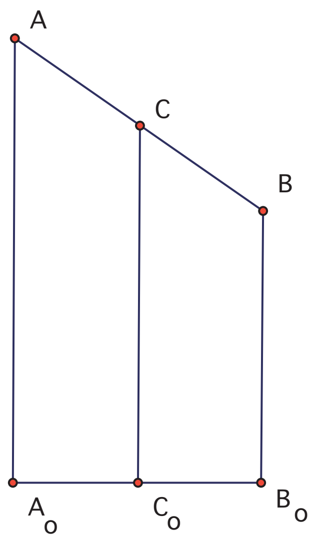

FIGURE t35a

Solution. Let AA_0 and BB_0 be the perpendiculars from points A and B to the given line, let C be the midpoint of segment AB, and draw $CC_0 \perp A_0B_0$. If points A and B lie on the same side of the given line (Figure t35a), then AA_0B_0B is a trapezoid. Line CC_0 is parallel to the bases of the trapezoid, so by Lemma 1, it must bisect A_0B_0. Therefore $CC_0 = \frac{1}{2}(AA_0 + BB_0)$.

If points A and B lie on opposite sides of the given line (Figure t35b), then AB and A_0B_0 are diagonals of trapezoid AA_0BB_0. By Lemma 2 above, we have $CC_0 = \frac{1}{2}(AA_0 - BB_0)$.

Notes. The two lemmas in this solution are interesting in their own right. In many geometry courses they are proved as theorems early on. Often, they are treated as special cases of a slightly more general statement: if a set of parallel lines cuts off equal segments along one transversal, then they cut off equal segments along any transversal. Hadamard treats this statement, in turn, as a special case of the fundamental theorem of **113**.

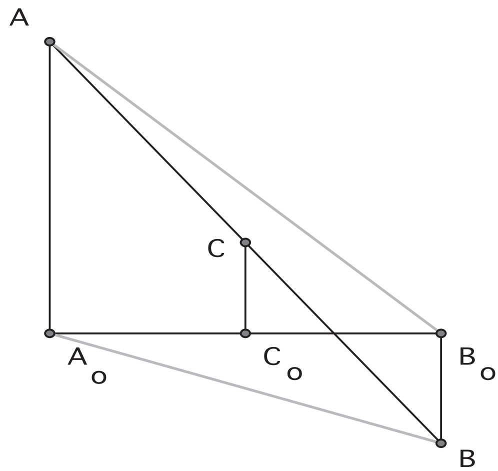

FIGURE t35b

Exercise 36. *The midpoints of the sides of any quadrilateral are the vertices of a parallelogram. The sides of this parallelogram are parallel to the diagonals of the given quadrilateral, and equal to halves of these diagonals. The center of the parallelogram is also the midpoint of the segment joining the midpoints of the diagonals of the given quadrilateral.*

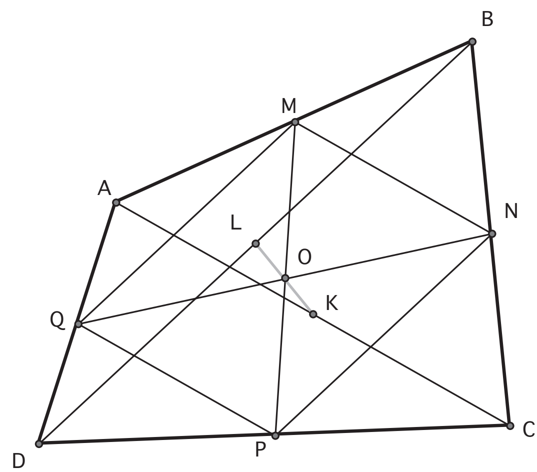

FIGURE t36

Solution. Let M, N, P, Q, K, L be the midpoints, respectively, of sides AB, BC, CD, DA and diagonals AC, BD of quadrilateral $ABCD$ (Figure t36). Segment MN, which joins two midpoints of triangle ABC, is parallel to AC and equal to half of it (**55**). For the same reason, segment PQ is parallel to AC and equal to half of it. Thus $PQ = MN$ and $PQ \parallel MN$. This shows (**46**, converse 2°) that quadrilateral $MNPQ$ is a parallelogram.

Lines MP and NQ bisect each other at their point O of intersection, since they are diagonals of a parallelogram. Segment KN, which joins two midpoints of triangle ABC, is parallel to AB and equal to half of it; similarly, LQ is parallel to AB and equal to half of it. It follows that $KNLQ$ is also a parallelogram, and that the midpoints of segments KL, NQ, which are diagonals of this new parallelogram, are both point O.

Note. Students can explore the possibility that K, N, L, Q might be collinear and thus fail to form a parallelogram.

Exercise 37. *Prove that the medians of a triangle are concurrent by extending the median CF (Figure 53 in the text) beyond F by a length equal to FG.*

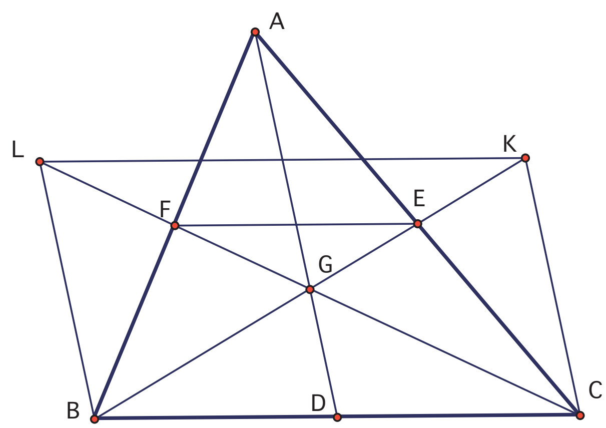

FIGURE t37

Solution. Suppose G is the intersection of medians BE and CF of triangle ABC (Figure t37), and choose points K and L so that $EK = EG$, $FL = FG$. Segment EF joints two midpoints both in triangle ABC and GKL, and so is parallel both to BC and KL and equal to half of either (**55**). It follows that segments BC and KL are equal and parallel, so $BCKL$ is a parallelogram. Therefore $BG = GK = 2GE$ and $CG = GL = 2GF$. Thus point G, on median BE, cuts off $\frac{1}{3}$ of segment CF.

In the same way, we can show that the third median AD cuts off $\frac{1}{3}$ of CF and therefore passes through the same point.

Exercise 38. *Given three lines passing through the same point O (all three distinct), and a point A on one of them, show that there exists:*

1°. *a triangle with a vertex at A and having the three lines as its altitudes (one exception);*

2°. *a triangle with a vertex at A and having the three lines as its medians;*

3°. *a triangle with a vertex at A and having the three lines as bisectors of its interior or exterior angles (one exception);*

4°. *a triangle with a midpoint of one of its sides at point A and having the three lines as perpendicular bisectors of the sides (reduce this to 1°).*

Solution. Let a, b, c be the given lines passing through O, and let A be the point on line a.

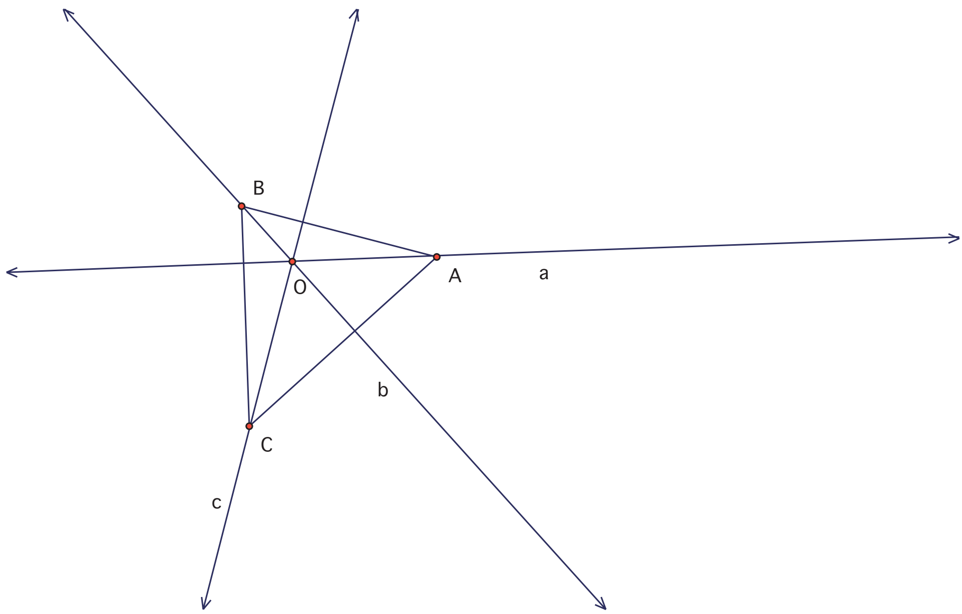

FIGURE t38a

1°. Sides AB and AC of a triangle ABC having its altitudes on the given lines must be perpendicular, respectively, to c and b (Figure t38a). This observation determines the positions of B and C on lines b and c. If lines b and c are perpendicular, then the lines through A perpendicular to b and c will clearly be parallel, respectively, to c and b, and the construction will not work. In this case, there is no triangle with its altitudes along the given lines and with the given point A as a vertex.

Notes. Students can be given the following hint, in the form of another problem.

Lines a, b, and c intersect at point O. Point A is chosen on line a. The perpendicular from A to c intersects b at point B. The perpendicular from B to a intersects c at point C. Prove that AC is perpendicular to b.

Students may give an original proof of this statement. Or they may notice that they are actually constructing a triangle together with its altitudes, and the fact that the altitudes are concurrent guarantees the conclusion.

But even with this sort of hint, this problem, and the others in this set, are not as easy to solve as may appear from a simple description of their solutions. Even if they cannot solve them independently, students can understand the solution, and learn something from it, by modeling the construction with dynamic geometry software. They can then vary the positions of the given elements (three lines and a point) to see what happens to the triangle they've constructed. Moving the given point along its line will simply shrink or expand the diagram. Rotating one of the lines around the given point distorts the triangle in more complicated ways, which students can explore, then explain.

Students' explorations can be motivated by a search for the exceptional case, when the construction fails. For example, students will discover that some rotations will move one vertex of the constructed triangle out of range of the figure. Rotating a bit more will give the triangle a "vertex at infinity", so that two of its "sides" are parallel. The dynamic software thus sheds light on the nature of the exceptional case.

Students may also discover other exceptional cases, not referred to in the solution. For example, if two of the given lines coincide, or if the point A coincides with the intersection of the three lines, the construction will fail. These "degenerate" cases of the situation are usually not worth mentioning. But in using geometry software, students often discover them quite naturally. In that case, it may be valuable to bring them into the discussion.

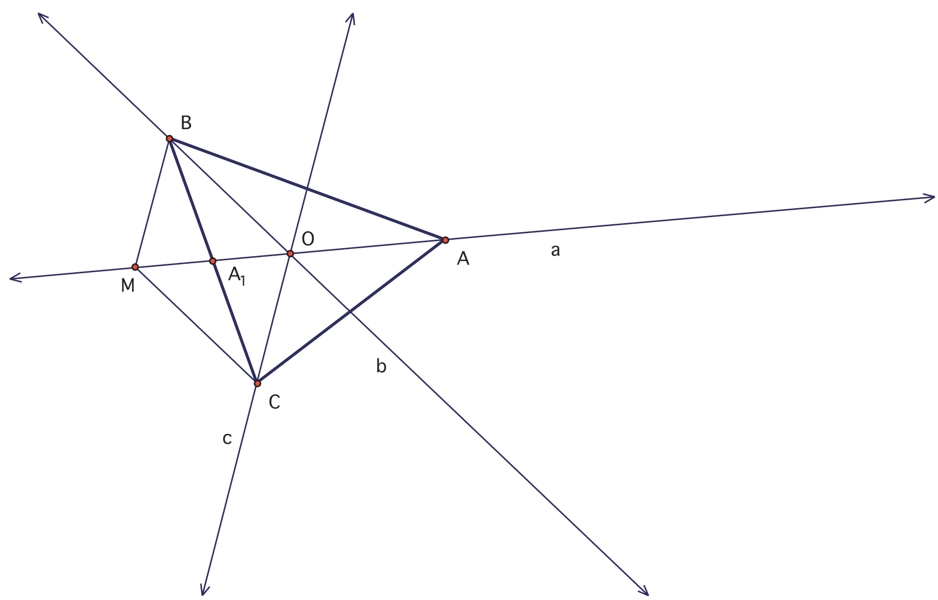

FIGURE t38b

2°. Suppose B and C are the other two vertices of the required triangle. We construct $OA_1 = \frac{1}{2}OA$ (Figure t38b), Then (since $A_1O = \frac{1}{3}A_1A$), O is the centroid (intersection point of the medians) of triangle ABC. We don't yet know where BC lies, but we know that A_1 must be its midpoint. If M is chosen on line a so that $A_1M = A_1O$, then $OBMC$ must be a parallelogram. This allows us to find the locations of points B and C by drawing lines through M parallel to c and b.

This construction, unlike the others, requires the introduction of a whole new figure—a parallelogram—into the discussion, and thus it is more difficult to think of than the others in this series. On the other hand, there are no exceptional cases. Students can verify this by experimenting with their software sketch.

3°. If lines a, b, c contain the angle bisectors of the required triangle ABC, then AC and BC must be symmetric with respect to line b, and AB and CB must be

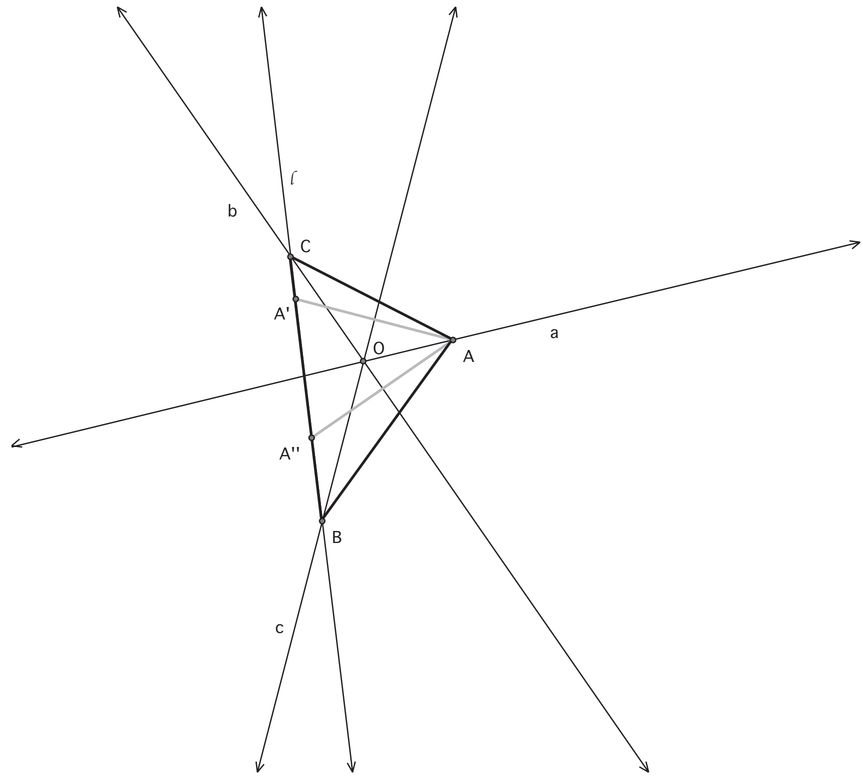

FIGURE t38c

symmetric with respect to line c (Figure t38c). It follows that the points symmetric to A with respect to b and c must lie on a line l passing through B and C. This allows us to construct line l, and therefore to find vertices B and C.

If lines b and c are perpendicular, then it is not hard to see that the line l passes through point O, and the problem has no solution. If line a is perpendicular to either b or c, then line l is parallel to the other one of them, and the problem again has no solution. In the other cases, that is, when no two of the given lines are perpendicular, the problem has a unique solution.

Students should be encouraged to explore this situation with geometry software. They will find that, for the same construction, some initial positions of the three lines will result in a solution with interior angle bisectors. For the very same construction, some positions will result in a triangle in which some of the original lines bisect the exterior angles. The intermediary situations are exactly the exceptional cases referred to in the problem statement.

4°. Consider triangle ABC, with its altitudes along lines a, b, and c (see 1°). Through its vertices, we can draw lines parallel to the opposite sides (see the proof of the theorem on the concurrence of altitudes, **53**). The triangle thus determined satisfies the conditions of the problem.

Exercise 39. *In a triangle, the larger side corresponds to the smaller median. (Consider the angle made by the third median with the third side.)*

A triangle with two equal medians is isosceles.

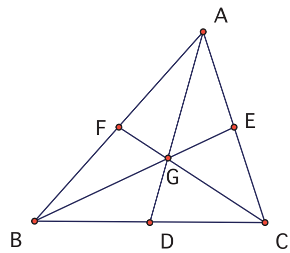

FIGURE t39

Solution. Let AD, BE, CF be the medians of triangle ABC (Figure t39), and let G be their point of intersection. If $AB > AC$, then triangles ABD and ACD have two pairs of equal sides and unequal third sides. Therefore (see the converse theorem in **28**), we have $\widehat{ADB} > \widehat{ADC}$. From triangles GBD and GCD it follows that $BG > GC$ (again from **28**); that is, $\frac{2}{3}BE > \frac{2}{3}CF$ and $BE > CF$.

In just the same way, we can show that if $AC > AB$, then $CF > BE$. It follows that if $BE = CF$, then $AB = AC$.

The logical strategy of proving the case for equality by proving the two cases for inequality is useful. Some students may find it subtle.

Exercise 40. *Let us assume that a billiard ball which strikes a flat wall will bounce off in such a way that the two lines of the path followed by the ball (before and after the collision) make equal angles with the wall.*

Consider n lines $D_1, D_2, \ldots, D_n$ in the plane, and points A, B on the same side of all of these lines. In what direction should a billiard ball be shot from A in order that it arrive at B after having bounced off each of the given lines successively? Show that the path followed by the ball in this case is the shortest broken line going from A to B and having successive vertices on the given lines.

SPECIAL CASE. *The given lines are the four sides of rectangle, taken in their natural order; the point B coincides with A and is inside the rectangle. Show that, in this case, the path traveled by the ball is equal to the sum of the diagonals of the rectangle.*

Solution. We present a solution for four lines D_1, D_2, D_3, D_4. (The construction would be similar if there were more lines.) Let $AXYZUB$ be the required broken line (Figure t40a). Assume that points X, Y, Z have already been found. We can find point U by joining point Z to B', the point symmetric to B with respect to line D_4 (Exercise 14). Thus we have reduced the problem to the construction of the path $AXYZB'$ for the case with only three lines D_1, D_2, D_3. In the same way, we can find point Z by joining Y to B'', the point symmetric to B' with respect to D_3, and so on.

Thus we have the following construction: We find point B', symmetric to B with respect to D_4, then point B'', symmetric to B' with respect to D_3, then point B''', symmetric to B'' with respect to D_2, and finally point $B^{(\text{iv})}$, symmetric to B''' with respect to D_1. The line $AB^{(\text{iv})}$ gives us point X, the line XB''' gives us point

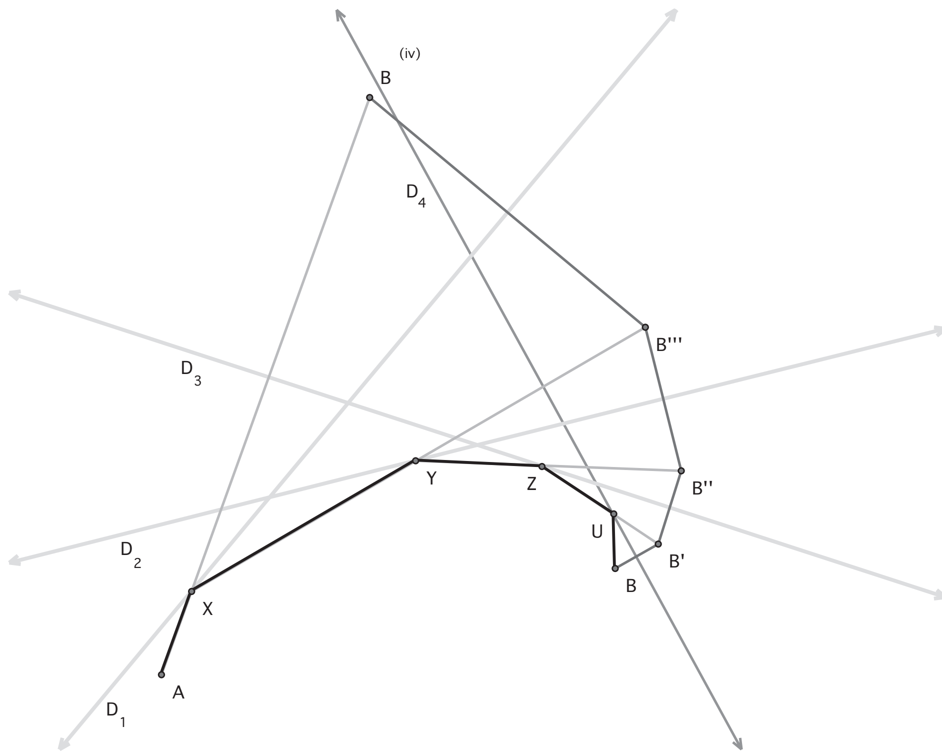

Figure t40a

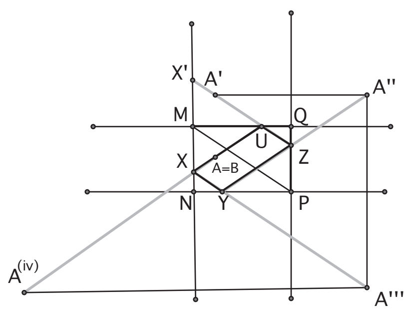

Figure t40b

Y, the line YB'' gives us point Z, and the line ZB' gives us point U. This method of construction does not depend on the number of lines given.

We now have equal paths $AXYZUB = AXYZB' = AXYB'' = AXB'''$, and these paths all equal $AB^{(\mathrm{iv})}$ (in length). The length of any other broken line path from A to B with vertices on lines D_1, D_2, D_3, D_4 will be equal to some broken line joining A and $B^{(\mathrm{iv})}$, and thus will be longer than $AB^{(\mathrm{iv})} = AXYZUB$.

The special case when D_1, D_2, D_3, D_4 are the sides of a rectangle and points A and B coincide is shown in Figure t40b. In this case, $XYZU$ is a parallelogram, and $MX = PZ = MX'$. It follows that $XU + UZ = X'Z = MP$, since $MPZX'$

is also a parallelogram. Thus $XY + YZ + ZU + UX = 2MP$ (the diagram makes clear which points are referred to).

Notes. It is important for students to perform the construction (or draw the diagram) themselves with different choices of lines. It is also useful for them to explain in detail how the discussion for the rectangle is a special case of the more general problem.

Another interesting special case involves a point on one side of a triangle (playing the role of A, which here coincides with B) and two lines which are the other two sides of the triangle. If we start with the foot of an altitude of the triangle, the path obtained, the so-called *orthic triangle* of the original triangle, is the triangle formed by the feet of the altitudes of the original triangle. This triangle has many interesting properties.

Exercise 41. *The diagonals of the two rectangles of Exercise* 26 *are situated on the same two lines, parallel to the sides of the given parallelogram (analogous to* **54***). One of these diagonals is equal to the difference and the other to the sum of the sides of the parallelogram.*

Solution. In Figure t26, point K lies on the bisectors of $\widehat{DAB}$ and $\widehat{ADC}$, and so is equidistant from lines AB and CD (see **36**). This last statement is also true of points M, Q, S. Therefore points K, M, Q, S all lie on the same line, which is the locus of points equidistant from lines AB and CD. In the same way, we can show that points L, N, P, R lie on the same line, which is the locus of points equidistant from lines BC and AD.

Now $SM = AB$, since $ABMS$ is a parallelogram, and $SK = MQ = BC$, since $AKDS$ and $BQCM$ are rectangles. It follows that $SQ = SM + MQ = AB + BC$, and (since SK, AD are diagonals of the same rectangle, and hence are equal), $KM = SM - SK = AB - AD$.

Notes. See also Exercises 26 and 109.

Exercise 42. *In an isosceles triangle, the sum of the distances from a point on the base to the other sides is constant.*

What happens if the point is taken on the extension of the base?

In an equilateral triangle, the sum of the distances from a point inside the triangle to the three sides is constant.

What happens when the point is outside the triangle?

Solution. Suppose point M lies on the base BC of isosceles triangle ABC (Figure t42a), and suppose MK, ML, MN are the perpendiculars from M to sides AB, AC and to altitude CF of the triangle. Then $\widehat{CMN} = \widehat{CBA} = \widehat{BCL}$, and right triangles CMN and MCL are congruent. It follows that $ML = CN$. Since $MKFN$ is a rectangle, we also have $MK = NF$. Therefore $ML + MK = CN + NF = CF$, and the sum of the distances from M to AB and AC is equal to altitude CF of the triangle. If point M' is taken on the extension of BC, for example, past point C, then a similar argument shows that $M'K' - M'L' = M'K' - M'N' = N'K' = CF$.

Now suppose point M lies inside equilateral triangle ABC (Figure t42b), and MP, MQ, MR are the perpendiculars from M to sides BC, AC, AB. We draw line B_1C_1 through M, parallel to BC, and drop perpendiculars B_1K, B_1L from B_1 to sides AC and BC. We can use the result of the first part of this problem twice: once

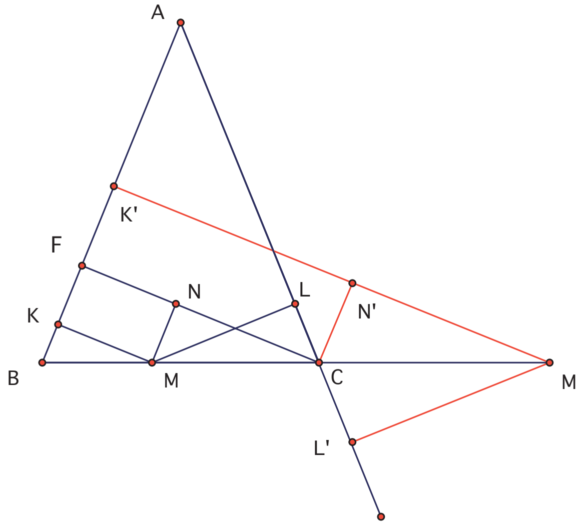

FIGURE t42a

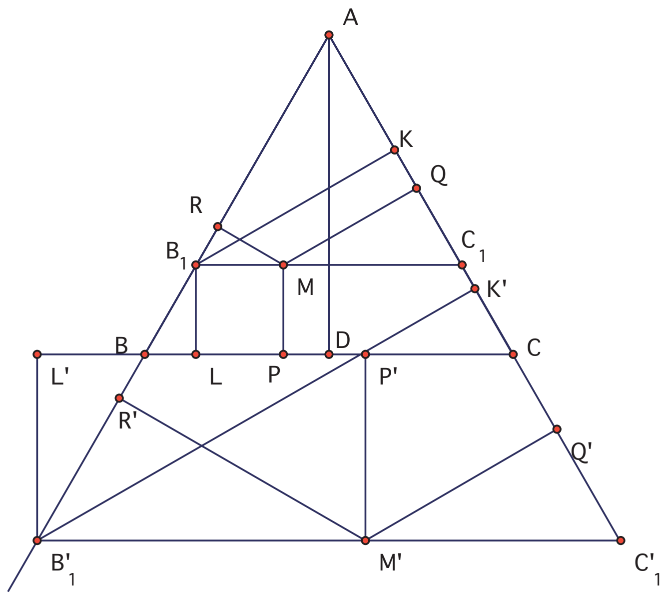

FIGURE t42b

for triangle AB_1C_1 and again for triangle ABC. If AD is an altitude of the original triangle, we find that $MQ + MR = B_1K$, $MP + MQ + MR = B_1L + B_1K = AD$ (since AD is equal to the altitude to AC from B in equilateral triangle ABC).

If point M' lies outside the triangle, for example in the region inside angle $\widehat{BAC}$, then $-M'P' + M'Q' + M'R' = -B_1'L' + B_1'K' = AD$. In any such case, we have $\pm MP \pm MQ \pm MR = AD$, with the a choice of sign depending on the region of the plane containing point M.

Notes. Students can learn from this rather sophisticated solution, even if they don't invent it themselves. One central point to be made is that an equilateral triangle is a special case of an isosceles triangle, and so results about an isosceles triangle can be applied to an equilateral triangle, with any side of the equilateral triangle playing the role of the base.

An alternative proof of this assertion can be constructed using concepts of area (see, for instance Exercise 298, where the problem is generalized).

Exercise 43. *In triangle ABC, we draw a perpendicular through the midpoint D of BC to the bisector of angle A. This line cuts off segments on the sides AB, AC equal respectively to $\frac{1}{2}(AB + AC)$ and $\frac{1}{2}(AB - AC)$.*

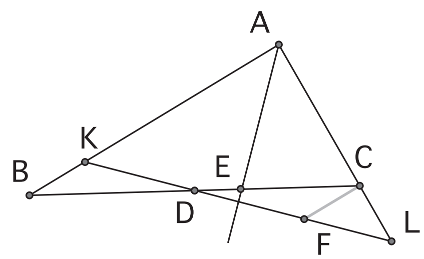

FIGURE t43

Solution. Suppose the triangle is ABC and is labeled so that $AB > AC$. Suppose the line perpendicular to angle bisector AE, passing through the midpoint D of side BC, intersects AB in K and AC in L (Figure t43). In triangle AKL, an angle bisector coincides with an altitude, so (Exercise 5) $AK = AL$ and $\widehat{AKL} = \widehat{ALK}$. If CF is now drawn parallel to AB, then triangles BDK and CDF are congruent, so $BK = CF$. Also, $\widehat{CFL} = \widehat{AKL} = \widehat{CLF}$, so $CF = CL$, and therefore $BK = CL$. So we have $AB = AK + BK$, $AC = AL - CL = AK - BK$, and thus $AK = AL = \frac{1}{2}(AB + AC)$ and $BK = CL = \frac{1}{2}(AB - AC)$.

Notes. This problem is deceptively simple to state, but difficult to solve. One difficulty comes from the "natural" way to draw the diagram, with one side of the original triangle parallel to an edge of the page on which it is drawn. In this position, the symmetry of isosceles triangle AKL, a key to the solution, is obscured (see Exercise 21 for a similar situation).

Once students get over this difficulty, they may see that $AK = AL$ and guess that $BK = CL$ (which would give them the result of the problem). The difficulty is that there are no congruent triangles that include these two segments: the introduction of $CF \parallel AB$ is another insight difficult for students to achieve. They might be helped by the hint to draw a segment equal to BK which is "closer" to CL. Some students may fortuitously try to translate BK to get CF, and then the proof will come more easily.

Students can investigate the situation in which the triangle is labeled so that $AB < AC$. They will find that the two segments in question are equal to $\frac{1}{2}(AB + AC)$ and $\frac{1}{2}(AC - AB)$. The problem might have been more precisely stated so as to assert that the second segment mentioned is $\frac{1}{2}|AB - AC|$.

Exercise 44. *Let $ABCD$, $DEFG$ be two squares placed side by side, so that sides DC, DE have the same direction, and sides AD, DG are extensions of each other. On AD and on the extension of DC, we take two segments AH, CK equal to DG. Show that quadrilateral $HBKF$ is also a square.*

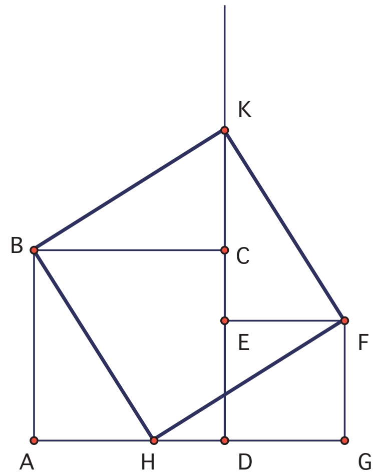

FIGURE t44

Solution. It is not hard to see that the triangles ABH, CBK, EKF, GHF (Figure t44) are all congruent, and therefore $HB = BK = KF = FH$. Angle $\widehat{BKF}$ is a right angle, since it is equal to the sum of the acute angles of right triangle BKC. This is enough to show that $HBKF$ is a square.

Notes. This problem is related to one of the proofs of the Pythagorean theorem (see Exercise 309). Students who already know the statement of this theorem can be invited to discover a proof of it, using Figure t44.

Indeed, the congruence of the four triangles noted above shows that the area of the large square is the sum of the areas of the small squares. It remains to note that the large square is drawn on the hypotenuse of one of these triangles, and the two smaller squares are drawn on the legs of these triangles (and one can reproduce this diagram "starting" with any right triangle at all).

Exercise 45. *On sides AB, AC of a triangle, and outside the triangle, we construct squares $ABDE$, $ACGF$, with D and F being the vertices opposite A. Show that:*

1°. *EG is perpendicular to the median from A and equal to twice this median;*

2°. *The fourth vertex I of the parallelogram with vertices EAG (with E and G opposite vertices) lies on the altitude from A in the original triangle;*

3°. *That CD, BF are equal and perpendicular to BI, CI, respectively, and their intersection point is also on the altitude from A.*

Lemma. *If two congruent triangles have the same sense of rotation and one pair of corresponding sides is perpendicular, then every other pair of corresponding sides is also perpendicular.*

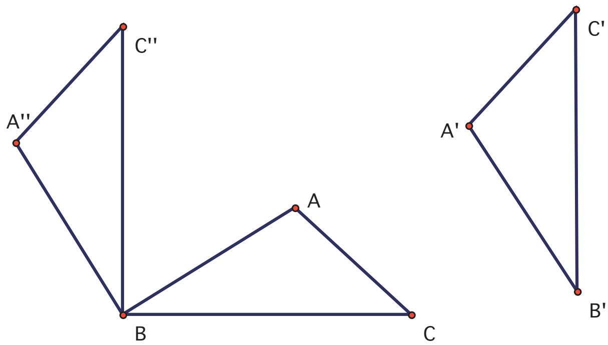

FIGURE t45a

Proof. Suppose triangles ABC, $A'B'C'$ are congruent with the same sense of rotation, and suppose side BC (Figure t45a) is perpendicular to side $B'C'$. Starting at B, we lay off segments BA'' and BC'' equal and parallel, respectively, to $B'A'$ and $B'C'$, and with the same orientation. Angles $\widehat{A''BC''}$ and $\widehat{A'B'C'}$ are then equal (**43**) with the same orientation. Since $\widehat{ABC}$ and $\widehat{A'B'C'}$ are equal by hypothesis with the same orientation, it follows that $\widehat{A''BC''}$ and $\widehat{ABC}$ are likewise equal with the same orientation. Therefore, $\widehat{C''BC}$ and $\widehat{A''BA}$ are equal, since they can be obtained by adding $\widehat{C''BA}$ to $\widehat{ABC}$ and $\widehat{A''BC''}$, respectively. Now BC'' is parallel to $B'C'$, and $B'C'$ is perpendicular to BC, so $\widehat{C''BC}$ is a right angle. Therefore, $\widehat{A''BA}$ is also a right angle. It follows that BA'' is perpendicular to AB, as is $B'A'$. In the same way, we can show that $A'C'$ is perpendicular to AC.

Solutions. 1°. We extend median AM of triangle ABC (Figure t45b) by its own length to K. (Note that $ABKC$ is a parallelogram, since its diagonals bisect each other.) Then $AB = AE$, $BK = AC = AG$, and $\widehat{ABK} = \widehat{EAG}$, since they have the same orientation and pairs of perpendicular sides. Thus triangles ABK and EAG are congruent with the same sense of rotation. It follows that $EG = AK = 2AM$, and our Lemma then tells us that EG is perpendicular to AK (since AB is perpendicular to EA).

2°. We first show that triangles EAI, ABC are congruent. Indeed, we have $AE = BA$ and $EI = AG = AC$. Now $\widehat{AEI}$ is supplementary to $\widehat{EAG}$ (they are consecutive angles in parallelogram $EAGI$), as is $\widehat{BAC}$ (since the four angles about point A add up to 360°, and two of them are right angles). Hence $\widehat{AEI} = \widehat{BAC}$, and the triangles are congruent.

These two triangles also have the same sense of rotation, and $EA \perp AB$, so $AI \perp BC$ (by our Lemma). This means that point I lies on the altitude of triangle ABC

3°. Triangles BCD and AIB are congruent with the same sense of rotation, and their sides BD and AB are perpendicular, so CD and BI are equal and perpendicular. In the same way, we can show that BF and CI are equal and perpendicular.

Finally, lines CD, BF, and AI all pass through the same point, since they are altitudes of triangle BCI.

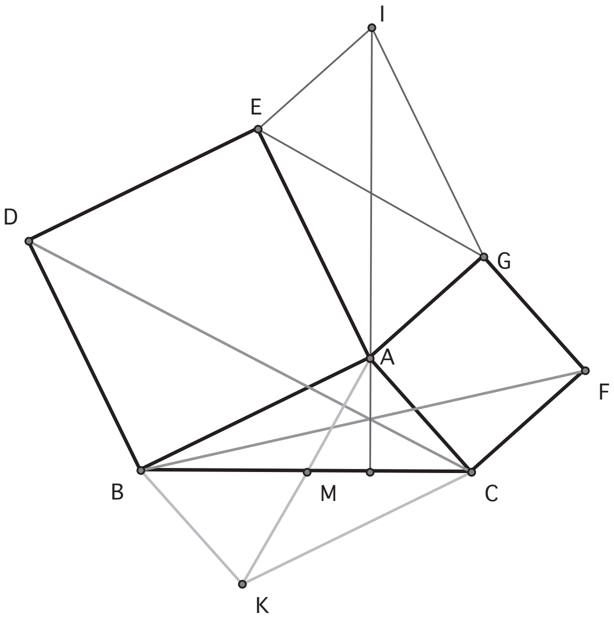

Figure t45b

Notes. The Lemma associated with this proof amounts to nothing less than a new way to use congruent triangles in certain situations. It can even be generalized: if two congruent triangles have the same sense of rotation, and one pair of corresponding sides (perhaps extended) form an angle α, then the other pairs of sides form the same angle α.

Exercise 46. *We are given a right angle $\widehat{AOB}$ and two perpendicular lines through a point P, the first intersecting the sides of the angle in A, B and the second intersecting the same sides in C, D. Show that the perpendiculars from points D, O, C to line OP intercept on AB segments equal to AP, PB, respectively, but situated in the opposite sense.*

Solution. In Figure t46, we must show that $XY = PA$ and $YZ = PB$. Suppose K is the midpoint of segment AD. Now in triangles APD, AOD, medians KP and KO are equal to half of the hypotenuse (**48**), so $KA = KP = KO = KD$. If we drop perpendiculars AA_1, KK_1, DD_1 to line OP from points A, K, D, respectively, then $OK_1 = K_1P$ (since $KO = KP$) and $D_1K_1 = K_1A_1$ (see Lemma 1 in the solution to Exercise 35), so $D_1O = PA_1$.

If X is the intersection of lines AB and DD_1, and $OY \perp OP$, $YU \perp DX$, then triangles XYU and APA_1 are congruent (since $UY = D_1O = PA_1$ and $\widehat{XYU} = \widehat{APA_1}$), so $XY = AP$.

Now suppose line CZ is perpendicular to OP. We can repeat this argument, with B in place of A, D in place of C, and Z in place of X (but with points P and Y retaining their roles), to show that $YZ = PB$.

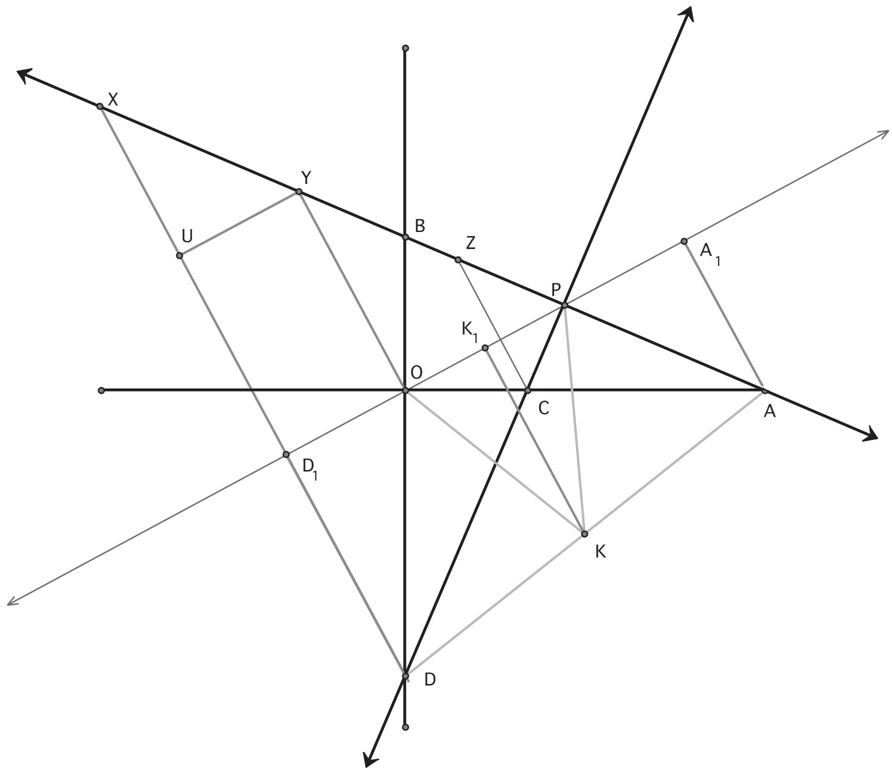

FIGURE t46