Preface

The purpose of this monograph is to present some recent directions of homogenization theory with particular emphasis on differential operators with rapidly oscillating coefficients, boundary value problems with rapidly alternating boundary conditions, equations in perforated domains, and other topics developed intensively during the last decades.

To explain the goal and ideas of homogenization theory for differential operators, we consider the boundary value problem for an equation with rapidly oscillating periodic coefficients:

$$-\operatorname{div}(A(x/\varepsilon)\nabla u_\varepsilon) = f \text{ in } \Omega, \quad u_\varepsilon = 0 \text{ on } \partial\Omega, \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $A(\xi)$ is a matrix with periodic measurable entries satisfying the ellipticity and boundedness conditions

$$\alpha I \le A \le \alpha^{-1} I, \quad \alpha > 0,$$

and I is the identity matrix.

The boundary value problem (0.1) simulates the simplest stationary processes in a strongly inhomogeneous medium, i.e., a medium whose characteristics may sharply change depending on the spatial variables and the parameter ε that characterizes, in a certain sense, the inhomogeneity scale. Such processes are quite difficult to study even with modern supercomputers because the use of finite-difference methods requires that the step be much less than ε, which leads to a huge amount of computations for small ε. It seems reasonable to try to replace (0.1) with another problem that is independent of small ε (i.e., it is the so-called "homogenized problem") and, at the same time, has a solution which is "close" to the solution of (0.1) for small ε. The question is how the "homogenized problem" and "closeness" should be understood. Answers are given by homogenization theory. We give a (nonrigorous) formulation of one of such homogenization results obtained first for a periodic inhomogeneous medium. There is a constant matrix $\widehat{A}$ whose entries depend only on the matrix $A(\xi)$ such that for any f the solutions $u_\varepsilon(x)$ converge to the solution of the problem

$$-\operatorname{div}(\widehat{A}\nabla u_0) = f \quad \text{in } \Omega, \quad u_0 = 0 \quad \text{on } \partial\Omega,$$

as $\varepsilon \to 0$; moreover, the entries of $\widehat{A}$ are determined by solving some auxiliary problem in the class of periodic functions.

The first proof of a homogenization theorem was obtained by De Giorgi and Spagnolo [**122, 123, 53, 54, 55**]. Shortly thereafter, Bakhvalov and then Lions established the same result by another method based on the asymptotic expansion technique, where a solution is represented in the form

$$u(x,\xi) + \varepsilon u_1(x,\xi) + \cdots, \quad \xi = x/\varepsilon,$$

with u and u_1 periodic with respect to ξ (cf. [**8, 9, 79**]). Using the technique of asymptotic analysis, it is possible not only to obtain a homogenized equation and establish the convergence of u_ε to the solution of the homogenized equation, but also to estimate the convergence rate. The asymptotic expansion method still remains one of the basic tools of homogenization theory.

Another approach to homogenization theory, based on the notion of compensated compactness, was developed by Murat [**91**] and Tartar [**124**].

In 1989, Nguetseng [**100**] introduced the notion of "two-scale convergence", which provides a new approach to homogenization problems. This approach, developed by Allaire [**3**], turns out to be very effective for problems of more complicated structure than in the case of a standard homogenization model. In particular, in applications there are homogenization problems where the solutions $u_\varepsilon(x)$ do not have a limit in the classical sense, but it can be proved that the norms $\|u_\varepsilon(x) - v(x, \frac{x}{\varepsilon})\|$ converge to zero in some function space. Here, the function $v(x, \xi)$ is periodic with respect to ξ and the variable x belongs to the domain Ω, whereas ξ belongs to a periodicity cell. In this situation, the u_ε do not have the classical limit, and the weak limit cannot be viewed as a satisfactory approximation of u_ε for small ε. However, the asymptotic behavior of the solution can be characterized by the so-called *two-scale* limit, i.e., a function $v(x, \xi)$ that is periodic with respect to ξ.

In applications, there are situations that could be characterized as the "partial oscillating behavior of $u_\varepsilon(x)$." For example, assume that Ω is divided into several domains (depending, in general, on ε) and the solution strongly converges to a limit in some of these domains, and oscillates in the remaining domains. This situation corresponds to materials or media consisting of several phases with quite different physical and mechanical properties of each phase (for example, a composite material consisting of a hard frame and a soft fill material; a medium consisting of a viscous compressible or incompressible fluid and elastic hard inclusions; a conducting material consisting of a well-conducting phase and a material whose properties are close to those of a dielectric). Then by a homogenized problem it is natural to understand a boundary value problem for a function $v(x, \xi)$ of two variables. Such a problem can be reduced to simpler problems of mathematical physics with unknown functions depending only on x or only on ξ, and the two-scale limit can be expressed as an algebraic combination of such functions. Such a procedure is usually referred to as the procedure of "asymptotic expansion." Recent research shows that an "asymptotic expansion" does not always hold even if the problem for the two-scale limit is well posed. However, the analysis of the homogenized problem can be informative even if the "asymptotic expansion" procedure is not applicable.

The literature on homogenization and related topics is quite extensive. We would like to draw the attention of the reader to the following books: [**4, 89, 85, 17, 50, 137, 138, 107, 102, 103, 10, 113, 6, 66, 67, 116, 115, 47, 44, 30, 15**]. Nevertheless, the authors run the risk of writing a new book with the hope that the presented material will be of interest for experts in homogenization theory as well as for nonexperts. This book was designed as an introduction to homogenization theory (so that modern methods are explained by rather simple examples and the rigorous proofs are accompanied by a number of exercises of different levels) and, at the same time, as a monograph combining little-known recent achievements with classical results of homogenization theory.

We list the main topics covered in this monograph.

Operators with random coefficients. Numerous works on homogenization of operators with random coefficients were published (cf., for example, [**137, 138**] and the references therein). In particular, for a random elliptic differential operator in the divergence form with stationary ergodic coefficients, the existence of a limit differential operator with constant deterministic coefficients was established and it was shown that the classical homogenization results remain valid for almost all realizations of a random medium (cf. [**137, 138**]). A remarkable observation concerning "nonclassical averaging" was made recently. Even if a homogenization procedure does not yield a deterministic limit operator, it is reasonable to study the limit behavior of the entire family of solutions in the sense of the weak convergence of measures regarded as solutions in a suitable function space. In such cases, the limit measure is usually determined by solving a stochastic partial differential equation.

Homogenization in p-connected domains. Recently, homogenization problems in p-connected domains (cf., for example, [**132**] and [**24**]) have been extensively studied. The "p-connectedness" is a generalization of the well-known notion of connectedness of a set, and it was introduced to describe processes in strongly inhomogeneous domains that are not connected from the geometric point of view, for example, electric current in a family of disconnected domains such that the intersection of the closures of these domains has positive capacity. Using the notion of p-connectedness, which is weaker than the usual geometric connectedness, it is possible to construct homogenization theory for differential operators in p-connected domains which is similar to homogenization theory in perforated domains in some ways but is quite different in other ways.

Appearance of a "term étrange." As was shown by Marchenko and Khruslov [**85**] in the 1960s, after homogenization of differential operators in perforated domains, a zeroth order term, called a "term étrange," can appear in the homogenized equation if the concentration (or the volume density) of holes tends to zero (this is similar to the appearance of a potential in the Schrödinger equation). This interesting phenomenon was investigated by many authors (cf., for example, [**45**]). We demonstrate the appearance of a "term étrange" by considering a simple example of the Dirichlet problem for the Laplace equation and then discuss other similar situations where a potential arises in the limit equation. We study in detail a problem with boundary conditions of the third type (the Fourier boundary conditions) on a perforation surface.

Boundary homogenization. One of the sections is devoted to the analysis of the behavior of solutions to boundary value problems in domains with oscillating boundary and to problems with rapidly oscillating type of boundary conditions (the so-called "boundary homogenization"). We pay particular attention to the method of matching asymptotic expansions, in spite of the fact that this method, suggested by Il′in [**69, 70**], is not directly related to the homogenization of differential operators and is well presented in the literature (cf., for example, [**71**]). However, recent research shows that the method of matching asymptotic expansions is an extremely effective tool in the study of boundary homogenization problems (cf., for example, [**60, 62, 63**]). Therefore, we find it reasonable to illustrate the main ideas of the matching method by a simple example and to show how the method works for boundary homogenization problems. In our opinion, this direction of homogenization theory is very promising for further developments.

Homogenization of equations of hydrodynamics. We consider a range of homogenization problems for equations of hydrodynamics, for example, the Darcy law relating pressure with speed of filtering fluid. To derive the Darcy law, we consider the boundary value problem for the Stokes equations with adherence condition on the boundary of cavities and then pass to a limit with respect to the characteristic size of inhomogeneity. This is done using two different methods. The first consists in passing to the limit in the corresponding integral identity and is similar to well-known classical homogenization methods. In the second case, the two-scale convergence technique is used. Unlike the weak limit, the two-scale limit is a function of two variables. To demonstrate features and advantages of the two-scale limit method, we start with a simple example of elliptic equations with rapidly oscillating coefficients and then proceed with the Stokes equations in a perforated domain.

Domains with rapidly oscillating boundary. To analyze the asymptotic behavior of solutions to boundary value problems in domains with rapidly oscillating boundary, for a model problem we take the initial-boundary value problem for a parabolic equation in a perforated domain with the boundary that periodically oscillates in time. An interesting phenomenon is observed in this case: first order terms describing translations appear in the limit equation although there are no translations in the original problem, which corresponds to the divergence form of the prelimit equation. To study this phenomenon, we introduce moving coordinates. Then the process can be described by an equation in the divergence form; this approach corresponds to the study of an effective diffusion process against the background of a large translation. The use of moving coordinates can also be helpful in the study of other problems, for example, the fluid filtration problem in domains with boundary alternating in time.

The book also contains various auxiliary information: the solvability theory of the main boundary value problems for the elasticity system and for the Stokes equations, some important inequalities, properties of function spaces, fundamental theorems in probability theory such as ergodic theorems, the central limit theorem for processes with mixing and the classical Bogolyubov averaging method.

The book is based on special lecture courses on modern aspects in mathematical physics, asymptotic methods, and homogenization of differential operators given by the authors at Lomonosov Moscow State University. The lectures were significantly revised and extended for this publication. The authors would like to express their deep gratitude to Professors V. V. Zhikov and P. P. Gadyl′shin for the extensive discussions and comments and, in particular, for the valuable additional information about the two-scale convergence, p-connectedness, and matching of asymptotic expansions.

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