CHAPTER 1

# Overview of our main results

In this chapter, we briefly summarize our homotopy colimit decompositions for the 26 sporadic groups $G$. The detailed statements of those results, and a great deal of further information, will be presented in Chapter 7. In each case there, we express the 2-completed classifying space $(BG)^\wedge_2$ as the completion of the homotopy colimit of a diagram of classifying spaces of subgroup of $G$. That diagram is a simplex, determined by an "incident" set of 2-local subgroups $H_i$ $(i \in I)$ and all their intersections $H_J := \bigcap_{i \in J} H_i$ for $\varnothing \neq J \subseteq I$; see Definition 4.6.2. The following table indicates, for each group $G$, the subgroups $H_i$ $(i \in I)$ used for this decomposition.

After the table, we will further discuss some of these diagrams of subgroups.

| Group | Subgroups |
|---|---|
| Mathieu group $M_{11}$ | $Q_8 : \Sigma_3$<br>$\Sigma_4$ |
| Mathieu group $M_{12}$ | $2_+^{1+4} : \Sigma_3$<br>$4^2 : D_{12}$ |
| Mathieu group $M_{22}$ | $2^4 : \mathrm{Alt}_6$<br>$2^3 : L_3(2)$<br>$2^4 : \Sigma_5$ |
| Mathieu group $M_{23}$ | $2^4 : \mathrm{Alt}_7$<br>$2^3 : L_3(2)$<br>$2^4 : (3 \times \mathrm{Alt}_5) : 2$ |
| Mathieu group $M_{24}$ | $2^4 : \mathrm{Alt}_8$<br>$2^6 : (L_3(2) \times \Sigma_3)$<br>$2^6 : 3 \cdot \Sigma_6$ |
| Janko group $J_1$ | $2^3 : 7 : 3$ |
| Janko group $J_2$ | $2_-^{1+4} : \mathrm{Alt}_5$<br>$2^{2+4} : (3 \times \Sigma_3)$ |
| Janko group $J_3$ | $2_-^{1+4} : \mathrm{Alt}_5$<br>$2^{2+4} : (3 \times \Sigma_3)$<br>$2^4 : (3 \times \mathrm{Alt}_5)$ |

| Group | Subgroups |
|---|---|
| Janko group $J_4$ | $2_+^{1+12}{:}3{}^{\cdot}M_{22}{:}2$ |
| | $2^{3+12}{}^{\cdot}\big(\Sigma_5 \times L_3(2)\big)$ |
| | $2^{10}{:}L_5(2)$ |
| | $2^{11}{:}M_{24}$ |
| Higman–Sims group $HS$ | $4{}^{\cdot}2^4{:}\Sigma_5$ |
| | $4^3{:}L_3(2)$ |
| | $2^4{:}\Sigma_6$ |
| McLaughlin group $McL$ | $2{}^{\cdot}\mathrm{Alt}_8$ |
| | $2^4{:}\mathrm{Alt}_7$ |
| | $2^4{:}\mathrm{Alt}_7$ |
| Suzuki group $Suz$ | $2_-^{1+6}{}^{\cdot}U_4(2)$ |
| | $2^{2+8}{:}(\mathrm{Alt}_5 \times \Sigma_3)$ |
| | $2^{4+6}{:}3{}^{\cdot}\mathrm{Alt}_6$ |
| Conway group $Co_3$ | $2{}^{\cdot}Sp_6(2)$ |
| | $2^{2+6}3^{1+2}2^2$ |
| | $2^4{}^{\cdot}L_4(2)$ |
| Conway group $Co_2$ | $2_+^{1+8}{:}Sp_6(2)$ |
| | $2^{4+10}(\Sigma_3 \times \Sigma_5)$ |
| | $(2_+^{1+6} \times 2^4)L_4(2)$ |
| | $2^{10}{:}M_{22}{:}2$ |
| Conway group $Co_1$ | $2_+^{1+8}{}^{\cdot}\Omega_8^+(2)$ |
| | $2^{2+12}{:}\big(\Sigma_3 \times L_4(2)\big)$ |
| | $2^{4+12}{}^{\cdot}\big(3{}^{\cdot}\Sigma_6 \times \Sigma_3\big)$ |
| | $2^{11}{:}M_{24}$ |
| Fischer group $Fi_{22}$ | $\big(2 \times 2_+^{1+8}{:}U_4(2)\big){:}2$ |
| | $2^{5+8}{:}(\Sigma_3 \times \mathrm{Alt}_6)$ |
| | $2^{10}{:}M_{22}$ |
| | $2^6{:}Sp_6(2)$ |
| Fischer group $Fi_{23}$ | $(2^2 \times 2_+^{1+8})\big(3 \times U_4(2)\big)2$ |
| | $2^{6+8}{:}(\Sigma_3 \times \mathrm{Alt}_7)$ |
| | $2^7{:}Sp_6(2)$ |
| | $2^{11}{}^{\cdot}M_{23}$ |

| Group | Subgroups |
|---|---|
| Fischer group $Fi'_{24}$ | $2^{1+12}_+ \cdot 3U_4(3)2$<br>$2^{3+12}\big(\mathrm{Alt}_6 \times L_3(2)\big)$<br>$2^{6+8}\big(\Sigma_3 \times L_4(2)\big)$<br>$2^8 : \Omega_8^-(2)$<br>$2^{11} \cdot M_{24}$ |
| Harada–Norton group $HN$ | $2^{1+8}_+(\mathrm{Alt}_5 \wr 2)$<br>$2^{3+2+6}\big(3 \times L_3(2)\big)$<br>$2^6 \cdot \Omega_6^-(2)$ |
| Thompson group $Th$ | $2^{1+8}_+ \cdot \mathrm{Alt}_9$<br>$2^5 \cdot L_5(2)$ |
| Baby Monster $B = F_2$ | $2^{1+22}_+ \cdot Co_2$<br>$2^{2+10+20}(M_{22} : 2 \times \Sigma_3)$<br>$2^3 2^{[32]}\big(\Sigma_5 \times L_3(2)\big)$<br>$2^{5+5+10+10} L_5(2)$<br>$2^{9+16} Sp_8(2)$ |
| Fischer–Griess Monster $M = F_1$ | $2^{1+24}_+ \cdot Co_1$<br>$2^{2+11+22}(M_{24} \times \Sigma_3)$<br>$2^{3+6+12+18}\big(3 \cdot \Sigma_6 \times L_3(2)\big)$<br>$2^{5+10+20}\big(\Sigma_3 \times L_5(2)\big)$<br>$2^{10+16} \cdot \Omega_{10}^+(2)$ |
| Held group $He$ | $2^{1+6}_+ L_3(2)$<br>$2^6 : 3 \cdot \Sigma_6$<br>$2^6 : 3 \cdot \Sigma_6$ |
| Rudvalis group $Ru$ | $2 \cdot 2^{4+6} : \Sigma_5$<br>$2^{3+8} : L_3(2)$<br>$2^6 : G_2(2)$ |
| O'Nan group $O'N$ | $4 \cdot L_3(4) : 2$<br>$4^3 \cdot L_3(2)$ |
| Lyons group $Ly$ | $2 \cdot \mathrm{Alt}_{11}$<br>$\big((2^{2+4} : (3 \times 3) : 2) \times 3\big) : 2$<br>$2^3 \cdot L_3(2)$<br>$(2^4 \times 3)\mathrm{Alt}_7$ |

Here is an illustration of a diagram arising from the subgroups in the table: in the case $G = M_{11}$, we have a diagram of groups

$$Q_8 : \Sigma_3 \longleftarrow D_8 \longrightarrow \Sigma_4$$

described by the two listed subgroups, and their (single) intersection. This gives us a diagram of classifying spaces

$$B(Q_8 : \Sigma_3) \longleftarrow BD_8 \longrightarrow B\Sigma_4$$

and the map from the homotopy colimit of this diagram to $BM_{11}$ is a mod 2 cohomology equivalence. This is equivalent to the statement that after Bousfield–Kan 2-completion, the map is a homotopy equivalence.

In each case, the diagram of groups is a simplex formed by taking the intersections of an "incident" (cf. Condition 6.2.7) set of representatives of the listed subgroups, and inclusions between these intersections. This simplex in fact arises as an orbit space, namely the quotient of a corresponding simplicial complex by a natural action of $G$; the result depends on the flag-transitivity (cf. Definition 6.4.2) of the group action, which is verified for each of the groups in turn.

In three of the 26 sporadic cases, we are able to reduce to a slightly smaller diagram than the simplex diagram. For $J_3$, we reduce the pushout cube to the following diagram:

$$2^{1+4}_{-} : \mathrm{Alt}_5 \longleftarrow 2^{2+4} : (3 \times 2) \longrightarrow 2^{2+4} : (3 \times \Sigma_3) \longleftarrow 2^{2+4} : 3^2 \longrightarrow 2^4 : (3 \times \mathrm{Alt}_5)$$

For $Fi'_{24}$ and $Ly$ we reduce to more complicated diagrams described in the relevant sections of Chapter 7.

In these three cases, the diagram can be thought of as being indexed by the set of simplices in a simplicial complex consisting of two simplices of the same dimension which share a common face of codimension one. For $J_3$ this complex is 1-dimensional, for $Ly$ it is 2-dimensional, and for $Fi'_{24}$ it is 3-dimensional. We picture these complexes below: