Preface

Random configurations of points in space, also known as point processes, have been studied in mathematics, statistics and physics for many decades. In mathematics and statistics, the emphasis has been on the Poisson process, which can be thought of as a limit of picking points independently and uniformly in a large region. Taking a different perspective, a finite collection of points in the plane can always be considered as the roots of a polynomial; in this coordinate system, taking the coefficients of the polynomial to be independent is natural. Limits of these random polynomials and their zeros are a core subject of this book; the other class consists of processes with joint intensities of determinantal form. The intersection of the two classes receives special attention, in Chapter 5 for instance. Zeros of random polynomials and determinantal processes have been studied primarily in mathematical physics. In this book we adopt a probabilistic perspective, exploiting independence whenever possible.

The book is designed for graduate students in probability, analysis, and mathematical physics, and exercises are included. No familiarity with physics is assumed, but we do assume that the reader is comfortable with complex analysis as in Ahlfors' text (1) and with graduate probability as in Durrett (20) or Billingsley (6). Possible ways to read the book are indicated graphically below, followed by an overview of the various chapters.

The book is organized as follows:

Chapter 1 starts off with a quick look at how zeros of a random polynomial differ from independently picked points, and the ubiquitous Vandermonde factor makes its first appearance in the book. Following that, we give definitions of basic notions such as point processes and their joint intensities.

Chapter 2 provides an introduction to the theory of Gaussian analytic functions (GAFs) and gives a formula for the first intensity of zeros. We introduce three important classes of geometric GAFs: planar, hyperbolic and spherical GAFs, whose zero sets are invariant in distribution under isometries preserving the underlying geometric space. Further we show that the intensity of zeros of a GAF determines the distribution of the GAF (Calabi's rigidity).

Chapter 3 We prove a formula due to Hammersley for computing the joint intensities of zeros for an arbitrary GAF.

Chapter 4 introduces determinantal processes which are used to model fermions in quantum mechanics and also arise naturally in many other settings. We show that general determinantal processes may be realized as mixtures of "determinantal projection processes", and use this result to give simple proofs of existence and central limit theorems. We also present similar results for permanental processes, which are used to model bosons in quantum mechanics.

Chapter 5 gives a deeper analysis of the hyperbolic GAF. Despite the many similarities between determinantal processes and zeros of GAFs, this function provides the only known link between the two fields. For a certain value of the parameter, the zero set of the hyperbolic GAF is indeed a determinantal process and this discovery allows one to say a great deal about its distribution. In particular, we give a simple description of the distribution of the moduli of zeros and obtain sharp asymptotics for the “hole probability" that a disk of radius r contains no zeros. We also obtain a law of large numbers and reconstruction result for the hyperbolic GAFs, the proofs of these do not rely on the determinantal property.
Chapter 6 studies a number of examples of determinantal point processes that arise naturally in combinatorics and probability. This includes the classical Ginibre and circular unitary ensembles from random matrix theory, as well as examples arising from non-intersecting random walks and random spanning trees. We give proofs that these point processes are determinantal.
Chapter 7 turns to the topic of large deviations. First we prove a very general result due to Offord which may be applied to an arbitrary GAF. Next we present more specialized techniques developed by Sodin and Tsirelson which can be used to determine very precisely, the asymptotic decay of the hole probability for the zero set of the planar GAF. The computation is more difficult in this setting, since this zero set is not a determinantal process.
Chapter 8 touches on two advanced topics, dynamical Gaussian analytic functions and allocation of area to zeros.

In the section on dynamics, we present a method by which the zero set of the hyperbolic GAF can be made into a time-homogeneous Markov process. This construction provides valuable insight into the nature of the repulsion between zeros, and we give an SDE description for the evolution of a single zero. This description can be generalized to simultaneously describe the evolution of all the zeros.

In the section on allocation, we introduce the reader to a beautiful scheme discovered by Sodin and Tsirelson for allocating Lebesgue measure to the zero set of the planar GAF. The allocation is obtained by constructing a random potential as a function of the planar GAF and then allowing points in the plane to flow along the gradient curves of the potential in the direction of decay. This procedure partitions the plane into basins of constant area, and we reproduce an argument due to Nazarov, Sodin and Volberg that the diameter of a typical basin has super-exponentially decaying tails.

The inter-dependence of the chapters is shown in Figure 1 schematically.

Acknowledgements

We are particularly grateful to Fedor Nazarov, Misha Sodin, Boris Tsirelson and Alexander Volberg for allowing us to reproduce their work here. Ron Peled, Misha Sodin, Tonci Antunovic and Subhroshekhar Ghosh gave us numerous comments and corrections to an earlier draft of the book. Many thanks also to Alexander Holroyd for creating the nice stable allocation pictures appearing in chapter 8. The second author would like to thank Microsoft research, SAMSI, and University of Toronto and U.C. Berkeley where significant portions of the book were written. In addition,

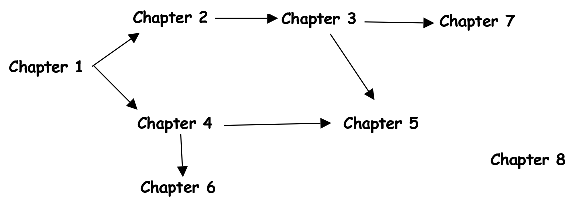

FIGURE 1. Dependence among chapters.

we thank the following people for their comments, discussions and suggestions: Persi Diaconis, Yogeshwaran Dhandapani, Jian Ding, Ning Weiyang, Steve Evans, Russell Lyons, Alice Guionnet, Ofer Zeitouni, Tomoyuki Shirai, Balázs Szegedy.